

# Chapter 1C–1D: Counting, Multiplication Rule, and Binomial Coefficients

## MATH/STAT 394: Probability I

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By the end of this lecture, students should be able to:

- ▶ apply the multiplication rule for counting;
- ▶ distinguish between sampling with and without replacement;
- ▶ compute permutations and ordered selections;
- ▶ compute combinations and unordered selections;
- ▶ understand the binomial coefficient combinatorially;
- ▶ recognize and avoid overcounting;
- ▶ prove combinatorial identities using counting arguments.

## The Multiplication Rule (Product Rule)

## Multiplication Rule

Suppose a procedure consists of two stages.

- ▶ Stage 1 can be done in **a** ways;
- ▶ for each choice in Stage 1, Stage 2 can be done in **b** ways.

Then the entire procedure can be done in

$$\mathbf{a \times b}$$

ways.

More generally, if a process has stages with

$$n_1, n_2, \dots, n_k$$

possible choices respectively, then the total number of outcomes is  $n_1 n_2 \cdots n_k$ .

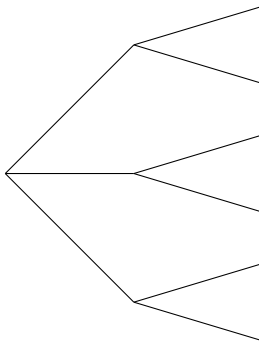
## Tree Interpretation

The multiplication rule can be visualized using a tree.

Suppose:

- ▶ there are 3 choices for the first step;
- ▶ each leads to 2 choices for the second step.

Then, according to the Multiplication Rule, there are  $3 \times 2 = 6$  possible outcomes.



## Example: License Plates

Suppose a license plate consists of 3 letters, followed by 4 digits. Assume repetition is allowed.



Figure: WA license plate. Source: Wikipedia.

Choices:

26 possibilities for each letter,

10 possibilities for each digit.

Thus the total number of possible license plates is

$$(26 \times 26 \times 26) \times (10 \times 10 \times 10 \times 10) = 26^3 \cdot 10^4 = 175,760,000 \text{ (?!)}$$

## Example: Binary Strings

A binary string of length  $n$  consists only of 0's and 1's.

Each position has:

2 choices.

Using the multiplication rule:

$$\underbrace{\boxed{\begin{array}{c} 2 \\ 0 \text{ or } 1 \end{array}}}_{\text{position 1}} \times \underbrace{\boxed{\begin{array}{c} 2 \\ 0 \text{ or } 1 \end{array}}}_{\text{position 2}} \times \cdots \times \underbrace{\boxed{\begin{array}{c} 2 \\ 0 \text{ or } 1 \end{array}}}_{\text{position } n} = 2^n.$$

Therefore, Number of binary strings of length  $n$  is  $2^n$ .

Remark: This is also equal to the number of all possible subsets of a set with  $n$  elements. (Why?)

## Sampling With and Without Replacement

## Ordered Sampling With Replacement

Suppose we have a set of size  $n$ , and we perform  $k$  draws with replacement, where  $1 \leq k \leq n$ .

**With replacement** means: after each draw, the selected object is returned before the next draw.

Therefore, each draw has exactly  $n$  possible choices.

Using the multiplication rule:

$$\underbrace{\boxed{n}}_{\text{choices}}_{\text{draw 1}} \times \underbrace{\boxed{n}}_{\text{choices}}_{\text{draw 2}} \times \cdots \times \underbrace{\boxed{n}}_{\text{choices}}_{\text{draw } k} = n^k.$$

Hence, the number of possible ways to draw  $k$  ordered items, with replacement, from a set of size  $n$ , is

$$n^k.$$

## Example: PIN Codes

A 4-digit PIN code uses digits 0–9.  
Digits may repeat.

Each position has:

10 choices.

Therefore:

$$10^4 = 10000$$

possible PINs.

## Ordered Sampling Without Replacement

Now suppose selected objects are *not* returned after each draw.

Then the number of available choices decreases after every selection.

Using the multiplication rule:

$$\underbrace{\boxed{n}}_{\text{choices}}_{\text{draw 1}} \times \underbrace{\boxed{n-1}}_{\text{choices}}_{\text{draw 2}} \times \underbrace{\boxed{n-2}}_{\text{choices}}_{\text{draw 3}} \times \cdots \times \underbrace{\boxed{n-k+1}}_{\text{choices}}_{\text{draw } k}$$
$$= n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

### Important

Without replacement means that the choices depend on previous selections.

## Example: Arranging Books

Suppose 5 distinct books are arranged on a shelf.

Choices:

5 choices for the first position,

4 for the second,

3 for the third,

etc.

Therefore, the number of arrangements is

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120.$$

## Factorials and Permutations

## Definition

For a positive integer  $n$ ,

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1.$$

Also define

$$0! = 1.$$

Examples:

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120.$$

A **permutation** is an ordered arrangement.

The number of ordered ways to choose  $k$  objects from  $n$  distinct objects is

$$n(n-1) \cdots (n-k+1).$$

Using factorial notation:

$$\frac{n!}{(n-k)!}$$

This is denoted by

$$P(n, k).$$

So, in summary

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1) \cdots (n-k+1).$$

## Example: Choosing Officers

Suppose a club has 12 members.

We choose:

- ▶ a president,
- ▶ a vice president,
- ▶ a secretary.

Since positions are different, order matters.

Therefore:

$$P(12, 3) = \frac{12!}{9!} = 12 \cdot 11 \cdot 10 = 1320.$$

# Combinations

Suppose we choose a committee of size  $k$  from  $n$  people.

Now:

$$\{A, B, C\}$$

and

$$\{B, A, C\}$$

represent the same committee.

Thus order does **not** matter.

This requires a different counting formula.

## Definition

The number of subsets of size  $k$  from a set of size  $n$  is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This is read as:

“ $n$  choose  $k$ ”.

Note:  $\binom{1}{1} = \binom{1}{0} = \binom{0}{0} = 1$ , can be justified by the formula or combinatorially.

## Why Divide by $k!$ ?

Suppose we first count ordered selections:

$$P(n, k) = \frac{n!}{(n-k)!}.$$

But each unordered subset is counted many times.

For example  $\{A, B, C\}$  can appear as:

$ABC, ACB, BAC, BCA, CAB, CBA.$

There are  $3! = 6$  orderings.

Therefore we divide by  $k!$ :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n, k)}{k!} \Rightarrow \binom{n}{k} \leq P(n, k).$$

## Example: Poker Hands

A poker hand consists of 5 cards from a 52-card deck.  
Order does not matter.

Thus the number of poker hands is

$$\binom{52}{5} = \frac{52!}{5!47!}.$$

## Example: Committees

How many committees of size 4 can be formed from 10 people?

Since order does not matter:

$$\binom{10}{4} = \frac{10!}{4!6!}.$$

Simplifying:

$$\binom{10}{4} = 210.$$

## Properties of Binomial Coefficients

## Identity

$$\binom{n}{k} = \binom{n}{n-k}$$

Reason:

Choosing  $k$  elements to include is equivalent to choosing  $n - k$  elements to exclude. :)

Algebraic Proof:

## Khayyam-Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

## Combinatorial Proof of Khayyam-Pascal's Identity

Suppose a committee of size  $k$  is selected from  $n$  people.

Focus on one distinguished person.

Two possibilities:

- 1 The distinguished person is not selected.

Then choose all  $k$  members from the remaining  $n - 1$  people:

$$\binom{n-1}{k}.$$

- 2 The distinguished person is selected.

Then choose the remaining  $k - 1$  members from the remaining  $n - 1$  people:

$$\binom{n-1}{k-1}.$$

Adding the two cases gives (why addition?)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

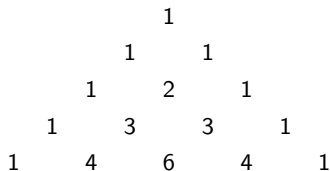
## Khayyam–Pascal Triangle: The Main Idea

The Khayyam–Pascal triangle is a triangular arrangement of numbers.

Each row begins and ends with 1.

Every interior entry is obtained by adding the two entries directly above it:

new entry = upper-left entry + upper-right entry

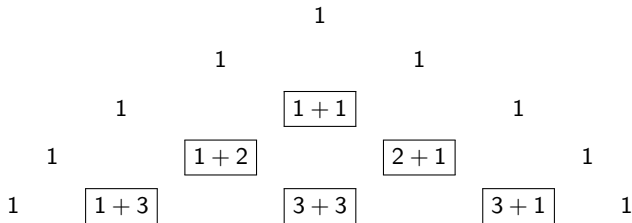


Source: Wikipedia: Pascal's Triangle.

# Building the Triangle

Start with 1 at the top.

Each new number is the sum of the two numbers above it.



# The Formula Inside the Triangle

|             |                |                  |                |                  |                    |                |                  |                  |                    |
|-------------|----------------|------------------|----------------|------------------|--------------------|----------------|------------------|------------------|--------------------|
| row 0       |                |                  |                |                  | $\binom{0}{0}$     |                |                  |                  |                    |
| row 1       |                |                  |                |                  | $\binom{1}{0}$     |                | $\binom{1}{1}$   |                  |                    |
| row 2       |                |                  |                | $\binom{2}{0}$   |                    | $\binom{2}{1}$ |                  | $\binom{2}{2}$   |                    |
| row 3       |                |                  | $\binom{3}{0}$ |                  | $\binom{3}{1}$     |                | $\binom{3}{2}$   |                  | $\binom{3}{3}$     |
|             |                |                  |                |                  | $\vdots$           |                |                  |                  |                    |
| row $n - 1$ |                | $\binom{n-1}{0}$ | $\dots$        |                  | $\binom{n-1}{k-1}$ |                | $\binom{n-1}{k}$ | $\dots$          | $\binom{n-1}{n-1}$ |
| row $n$     | $\binom{n}{0}$ | $\dots$          |                | $\binom{n}{k-1}$ |                    | $\binom{n}{k}$ |                  | $\binom{n}{k+1}$ | $\dots$            |

Each interior entry is obtained by adding the two entries directly above it:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

This identity is known as **Khayyam-Pascal's Rule**.

## Khayyam-Pascal's Rule/Identity

The rule for building the triangle is

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for  $1 \leq k \leq n-1$ .

For example,

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6.$$

So every interior entry is obtained by adding the two entries above it.

The rows of the triangle give the coefficients in binomial expansions.

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

- ▶ The coefficients are exactly the rows of the Khayyam–Pascal triangle.
- ▶ Hence, we can show that  $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$  (a very useful identity in Probability theory).
- ▶ Also proves why  $2^n = \sum_{i=0}^n \binom{n}{i}$ .  
Hint:  $2 = 1 + 1$  :)

## Story Proofs and Overcounting

A **story proof** proves an identity by showing that both sides count the same object in different ways.

Advantages:

- ▶ gives intuition;
- ▶ explains why identities are true;
- ▶ often simpler than algebraic proofs.

### Identity

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Why?

The left-hand side counts subsets of an  $n$ -element set by grouping according to subset size.

The right-hand side counts subsets directly (see slide 60):  
each element is either:

included or not included.

Thus there are

$$2^n$$

subsets total.

- 1 How many binary strings of length 8 are there?
- 2 How many 3-letter strings can be formed from the English alphabet if repetition is allowed?
- 3 How many ordered arrangements of 4 people can be selected from 10 people?
- 4 How many committees of size 3 can be formed from 8 people?

### Answers

$$2^8 = 256, \quad 26^3, \quad \frac{10!}{6!}, \quad \binom{8}{3} = 56.$$

## Summary

- ▶ Number of ordered length- $k$  sequences from  $n$  options (repetition allowed):

$$n^k.$$

- ▶ Number of ordered length- $k$  sequences from  $n$  distinct objects (no repetition):

$$\frac{n!}{(n-k)!}.$$

- ▶ Number of permutations of  $k$  objects chosen from  $n$ :

$$P(n, k) = \frac{n!}{(n-k)!}.$$

- ▶ Number of combinations of  $k$  objects chosen from  $n$ :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- ▶ Fundamental identities for binomial coefficients:

$$\binom{n}{k} = \binom{n}{n-k},$$

and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

### **Temporary page!**

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