Lecture 2 (June 24)

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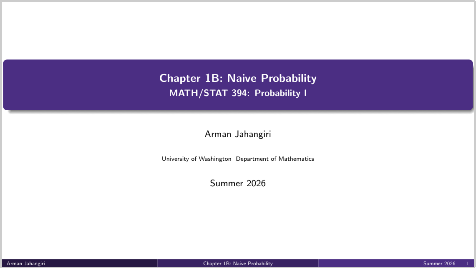

Chapter 1B: Naive Probability
MATH/STAT 394: Probability I

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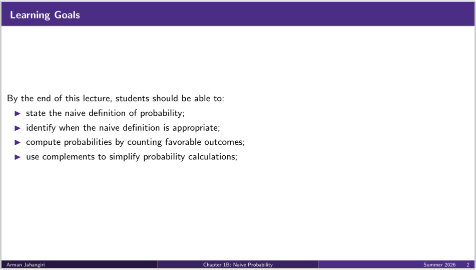
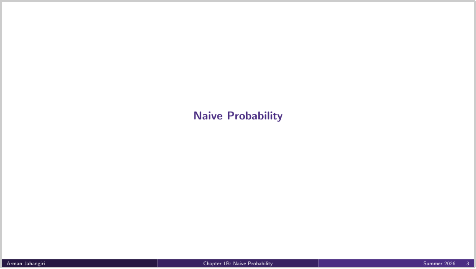


Learning Goals

By the end of this lecture, students should be able to:

- state the naive definition of probability;
- identify when the naive definition is appropriate;
- compute probabilities by counting favorable outcomes;
- use complements to simplify probability calculations;



Naive Probability

From Events to Probabilities

$f(x) = x^2$ $\mathbb{R} \to \mathbb{R}^{\geq 0}$ ①

S : sample space

$A \subseteq S$

$A \in P(S)$

In the previous lecture:

event

$\mathcal{S} = \{\text{all possible outcomes}\}, \quad A \subseteq \mathcal{S}.$

Now we want to assign a number to an event:

$\mathbb{P} : P(A) \to [0,1]$

$\mathbb{P}(A) =$ probability that A occurs.

$A \mapsto [0,1]$

Question

If every outcome in a finite sample space is equally likely, how should we define $\mathbb{P}(A)$?

$S = \{1,2,3\} \Rightarrow P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \underbrace{\{1,2,3\}}_{S}\}$

$\mathbb{P}(\{1\})$ $\cancel{P(1)}$ $\mathbb{P}(\{1,2\})$

Definition: Naive Probability

example) roll a die $S = \{1,2,\dots,6\}$ $|S| = 6$

$P(\{1,2,3\}) = \frac{|\{1,2,3\}|}{|S|} = \frac{3}{6} = \frac{1}{2}$

Naive Definition of Probability

Let A be an event for an experiment with a finite sample space $\mathcal{S}$.
If all outcomes in $\mathcal{S}$ are equally likely, define

$$\mathbb{P}_{\text{naive}}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{\text{number of outcomes favorable to } A}{\text{total number of outcomes in } \mathcal{S}}.$$

Here $|A|$ denotes the number of elements in the set A (aka *card*(A)).

Question: Why does this definition is suitable for equally-likely outcomes?

Naive Probability - Properties

Naive probability satisfies

$$0 \leq \mathbb{P}(A) \leq 1, \ \mathbb{P}(\mathcal{S}) = 1, \ \mathbb{P}(\emptyset) = 0, \qquad \text{for every } A \subseteq \mathcal{S}.$$

Why? First, if $A \subseteq \mathcal{S}$, then

$$0 \leq |A| \leq |\mathcal{S}|,$$

so $|A| \leq |S|$ $\Rightarrow \frac{|A|}{|S|} \leq 1$

$$0 \leq \frac{|A|}{|\mathcal{S}|} \leq 1. \checkmark$$

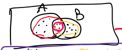
$E = A \cup B$

$E = A \cap B$

Also,

$$\mathbb{P}(\mathcal{S}) = \frac{|\mathcal{S}|}{|\mathcal{S}|} = 1, \checkmark \qquad \mathbb{P}(\emptyset) = \frac{0}{|\mathcal{S}|} = 0. \checkmark$$

Additivity for Disjoint Events



Suppose $A, B \subseteq \mathcal{S}$,
Then, since

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

$|A \cup B| = |A| + |B| - |A \cap B|$

naive def. of prob.

Therefore,

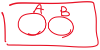
$$\mathbb{P}(A \cup B) = \frac{|A \cup B|}{|\mathcal{S}|} = \frac{|A| + |B| - |A \cap B|}{|\mathcal{S}|} = \frac{|A|}{|\mathcal{S}|} + \frac{|B|}{|\mathcal{S}|} - \frac{|A \cap B|}{|\mathcal{S}|}.$$

Thus

$$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}.$$

A speical case is when A and B are exclusive ($A \cap B = \emptyset$), when we have

$$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)} \checkmark$$



$|A| + |B|$

Looking ahead

Later, we will define probability more generally using properties like these as axioms. Naive probability will then become a special case of the general theory.

What the Definition Means

The naive definition says:

$$\mathbb{P}(A) = \frac{\text{how many outcomes make } A \text{ happen}}{\text{how many outcomes are possible}}.$$

✓

This requires two assumptions:

1. The sample space $\mathcal{S}$ is finite (we can't divide by infinity).
2. All outcomes in $\mathcal{S}$ are equally likely.

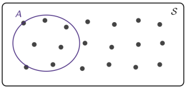
Important

The naive definition is not wrong. It is just limited. It should only be used when its assumptions are justified.

Pebble World Interpretation

Suppose each pebble has the same mass.

$$\mathbb{P}(A) = \frac{\text{number of pebbles in } A}{\text{number of pebbles in } \mathcal{S}}.$$



$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|}.$$

Example: One Fair Die

Roll one fair six-sided die.

$$\mathcal{S} = \{1,2,3,4,5,6\}.$$

Let

$$A = \{\text{roll is even}\} = \{2,4,6\}.$$

Then

$$|\mathcal{S}| = 6, \qquad |A| = 3.$$

Therefore

$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{3}{6} = \frac{1}{2}.$$

Example: One Card

Draw one card from a standard well-shuffled 52-card deck.
Let

$$A = \{\text{card is an ace}\}.$$

There are 4 aces and 52 cards total, so

$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{4}{52} = \frac{1}{13}.$$

Let

$$H = \{\text{card is a heart}\}.$$

Then

$$\mathbb{P}(H) = \frac{13}{52} = \frac{1}{4}.$$

Example: Red or Ace

Draw one card from a standard 52-card deck.
Let

and

$$R = \{\text{card is red}\}, \qquad A = \{\text{card is an ace}\}.$$

We want

$$\mathbb{P}(R \cup A).$$

intersection of R & A

There are:

26 red cards, 4 aces, 2 red aces.

So

$$|R \cup A| = 26 + 4 - 2 = 28.$$

Therefore

$$\mathbb{P}(R \cup A) = \frac{28}{52} = \frac{7}{13}.$$

$$P(R\cup A) = P(R) + P(A) - P(A \cap R)$$

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{7}{13}$$

Complements

Complement Rule Under the Naive Definition

Let $A \subseteq \mathcal{S}$.
The complement of A is

$$A^c = \{s \in \mathcal{S} : s \notin A\}.$$

Since every outcome is either in A or not in A,

$$|A^c| = |\mathcal{S}| - |A|.$$

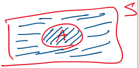
Therefore,

$$\mathbb{P}(A^c) = \frac{|A^c|}{|\mathcal{S}|} = \frac{|\mathcal{S}| - |A|}{|\mathcal{S}|} = 1 - \frac{|A|}{|\mathcal{S}|}.$$

So

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

$P(A^c) =$ in terms of $P(A)$



$A \subseteq S$

$A^c = S - A$

$A \cup A^c = S$

$$P(A^c) = 1 - P(A) \Longleftarrow \boxed{1 =} P(S) = P(A \cup A^c) = \boxed{P(A) + P(A^c)} - P(\underbrace{A \cap A^c}_{\emptyset})$$

Why Complements Are Useful

Sometimes it is easier to count when an event does **not** happen.

Instead of computing $\mathbb{P}(A)$ directly, compute:

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c).$$

Strategy

When a probability asks for phrases like

if you see these keywords

"at least one", "at least two", "not all",

first ask whether the complement is easier.

then think about computing both $P(A)$ & $P(A^c)$.
And then compute the one that is easier.

Example: At Least One Head

$S = \{(s_1, s_2, s_3, s_4, s_5) : s_i = H \text{ or } s_i = T\}$ $\boxed{|S| = 2^5}$

Flip a fair coin 5 times.
Let

$$A = \{\underline{\text{at least one head occurs}}\}.$$

let X = number of heads
then A says that $\{X \geq 1\}$

The complement is

$$A^c = \{\text{no heads occur}\}.$$

$\Rightarrow A^c : \{X < 1\}$

$\Rightarrow A^c : \{X = 0\}$ (no heads or all tails)

There is only one outcome with no heads: $TTTTT$.
The total number of outcomes is $2^5 = 32$.
Therefore

$$\mathbb{P}(A^c) = \frac{1}{32}, \qquad \mathbb{P}(A) = 1 - \frac{1}{32} = \frac{31}{32}.$$

$p(A)$

Example: At Least One Six

(at home)

Roll a fair die 4 times.
Let

$$A = \{\text{at least one six occurs}\}.$$

It is easier to count

$$A^c = \{\text{no six occurs}\}.$$

Total outcomes: 6^4. Outcomes with no six: 5^4.
Therefore,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{5^4}{6^4}.$$

Common Mistakes

Example: Two Fair Dice

Roll two fair dice.
A correct sample space is

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \cdots & (1,6) \\ (2,1) & (2,2) & & \cdots & (2,6) \\ (6,1) & (6,2) & & \cdots & (6,6) \end{bmatrix}$$

$$\mathcal{S} = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}.$$

Here (i,j) means:

$$i = \text{result of die 1}, \qquad j = \text{result of die 2}.$$

let X = first die
Y = second die

Thus

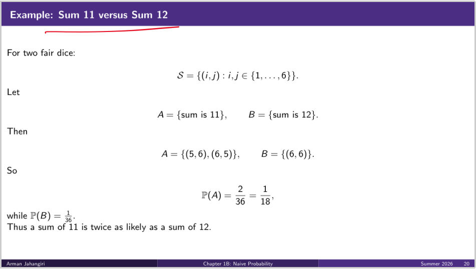
$$|\mathcal{S}| = 36. = 6^2$$

$P(X+Y=11) = \frac{2}{36} = \frac{1}{18}$

$P(X+Y=12) = \frac{1}{36}$

Important

The outcomes are ordered pairs. For example, $(5,6)$ and $(6,5)$ are different outcomes.



Example: Sum 11 versus Sum 12

For two fair dice:

$$\mathcal{S} = \{(i,j) : i,j \in \{1,\dots,6\}\}.$$

Let

$$A = \{\text{sum is } 11\}, \qquad B = \{\text{sum is } 12\}.$$

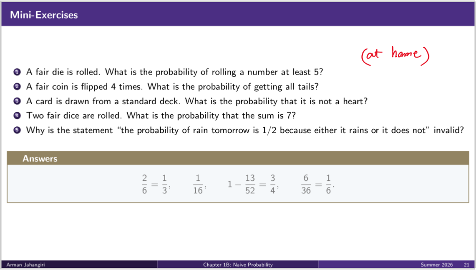
Then

$$A = \{(5,6),(6,5)\}, \qquad B = \{(6,6)\}.$$

So

$$\mathbb{P}(A) = \frac{2}{36} = \frac{1}{18},$$

while $\mathbb{P}(B) = \frac{1}{36}$.
Thus a sum of 11 is twice as likely as a sum of 12.



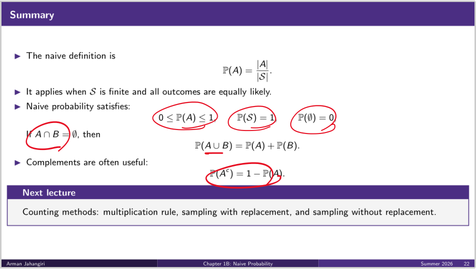
Mini-Exercises

(at home)

1. A fair die is rolled. What is the probability of rolling a number at least 5?
2. A fair coin is flipped 4 times. What is the probability of getting all tails?
3. A card is drawn from a standard deck. What is the probability that it is not a heart?
4. Two fair dice are rolled. What is the probability that the sum is 7?
5. Why is the statement "the probability of rain tomorrow is 1/2 because either it rains or it does not" invalid?

Answers

$$\frac{2}{6} = \frac{1}{3}, \quad \frac{1}{16}, \quad 1 - \frac{13}{52} = \frac{3}{4}, \quad \frac{6}{36} = \frac{1}{6}.$$



Summary

- The naive definition is
$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|}.$$
- It applies when $\mathcal{S}$ is finite and all outcomes are equally likely.
- Naive probability satisfies:
$$0 \le \mathbb{P}(A) \le 1, \qquad \mathbb{P}(\mathcal{S}) = 1, \qquad \mathbb{P}(\emptyset) = 0.$$
If $A \cap B = \emptyset$, then
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$
- Complements are often useful:
$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Next lecture

Counting methods: multiplication rule, sampling with replacement, and sampling without replacement.

chapter1cd_counting

Chapter 1C–1D: Counting, Multiplication Rule, and Binomial Coefficients

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Learning Goals

By the end of this lecture, students should be able to:

- apply the multiplication rule for counting;
- distinguish between sampling with and without replacement;
- compute permutations and ordered selections;
- compute combinations and unordered selections;
- understand the binomial coefficient combinatorially;
- recognize and avoid overcounting;
- prove combinatorial identities using counting arguments.

The Multiplication Rule (Product Rule)

The Basic Counting Principle

Multiplication Rule

Suppose a procedure consists of two stages.

- Stage 1 can be done in **a** ways;
- for each choice in Stage 1, Stage 2 can be done in **b** ways.

Then the entire procedure can be done in

$$\mathbf{a} \times \mathbf{b}$$

ways.

More generally, if a process has stages with

$$n_1, n_2, \ldots, n_k$$

possible choices respectively, then the total number of outcomes is $\boxed{n_1 n_2 \cdots n_k.}$

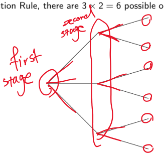
Tree Interpretation

The multiplication rule can be visualized using a tree.

Suppose:

- there are 3 choices for the first step;
- each leads to 2 choices for the second step.

Then, according to the Multiplication Rule, there are $3 \times 2 = 6$ possible outcomes.



$3 \times 2 = 6$

Example: License Plates

Suppose a license plate consists of 3 letters, followed by 4 digits. Assume repetition is allowed.

WASHINGTON
11
W 2025
CPU4704
EVERGREEN STATE

Figure: WA license plate. Source: Wikipedia.

$$\underbrace{26 \cdot 26 \cdot 26}_{3 \text{ letters}} \cdot \underbrace{10 \cdot 10 \cdot 10 \cdot 10}_{4 \text{ digits}}$$

Choices:

26 possibilities for each letter,

10 possibilities for each digit.

Thus the total number of possible license plates is

$$(26 \times 26 \times 26) \times (10 \times 10 \times 10 \times 10) = 26^3 \cdot 10^4 = 175,760,000 \quad (?!)$$

✓

Example: Binary Strings

100111010
a binary string of size ⑨

A binary string of length n consists only of 0's and 1's.

Each position has:

2 choices.

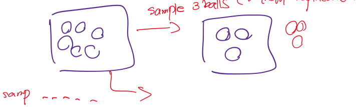
Using the multiplication rule:

$$\underbrace{\underset{0 \text{ or } 1}{\boxed{2}}}_{\text{position } 1} \times \underbrace{\underset{0 or 1}{\boxed{2}}}_{\text{position } 2} \times \cdots \times \underbrace{\underset{0 or 1}{\boxed{2}}}_{\text{position } n} = 2^n.$$

Therefore, Number of binary strings of length n is 2^n.
Remark: This is also equal to the number of all possible subsets of a set with n elements. (Why?)

Sampling With and Without Replacement

sample 3 balls (without replacement)



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Ordered Sampling With Replacement

Suppose we have a set of size n, and we perform k draws with replacement, where $1 \le k \le n$.

With replacement means: after each draw, the selected object is returned before the next draw.

Therefore, each draw has exactly n possible choices.

Using the multiplication rule:

$$\underbrace{\boxed{n}\ \text{choices}}_{\text{draw 1}} \times \underbrace{\boxed{n}\ \text{choices}}_{\text{draw 2}} \times \cdots \times \underbrace{\boxed{n}\ \text{choices}}_{\text{draw } k} = n^k.$$

Hence, the number of possible ways to draw k ordered items, with replacement, from a set of size n, is

$$n^k.$$

Example: PIN Codes

A 4-digit PIN code uses digits 0–9.
Digits may repeat.

Each position has:

$$10 \text{ choices.}$$

Therefore:

$$10^4 = 10000$$

possible PINs.

Ordered Sampling Without Replacement

for any $n \in \mathbb{N}$, $n! = n \times (n-1) \times \cdots \times 1$

Gamma

Now suppose selected objects are *not* returned after each draw.

Then the number of available choices decreases after every selection.

$n-(k-1)$

Using the multiplication rule:

$$\underbrace{\boxed{n}\ \text{choices}}_{\text{draw 1}} \times \underbrace{\boxed{n-1}\ \text{choices}}_{\text{draw 2}} \times \underbrace{\boxed{n-2}\ \text{choices}}_{\text{draw 3}} \times \cdots \times \underbrace{\boxed{n-k+1}\ \text{choices}}_{\text{draw } k}$$

$$= n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}. = \frac{n\cdot(n-1)\cdots(n-k+1)\,\overset{\cancel{(n-k)!}}{(n-k)\cdots 1}}{\cancel{(n-k)!}}$$

Important

Without replacement means that the choices depend on previous selections.

Example: Arranging Books

Suppose 5 distinct books are arranged on a shelf.

Choices:

5 choices for the first position,

4 for the second,

3 for the third,

etc.
Therefore, the number of arrangements is

$n=5$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120.$$

Factorials and Permutations

Factorials

Definition

For a positive integer n,

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

Also define

$$\boxed{0! = 1.}$$

Examples:

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120.$$

Permutations

A **permutation** is an ordered arrangement.

The number of ordered ways to choose $\boxed{k}$ objects from $\boxed{n}$ distinct objects is

$$\boxed{n(n-1)\cdots(n-k+1).}$$

Using factorial notation:

$$\|$$

$$\boxed{\frac{n!}{(n-k)!}}$$

This is denoted by

$$\|$$

$$P(n,k).$$

So, in summary

$$\boxed{P(n,k) = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1).}$$

Example: Choosing Officers

Suppose a club has $\boxed{12}$ members.
We choose:

- a president,
- a vice president,
- a secretary.

Since positions are different, order matters.

Therefore:

$$P(12,3) = \frac{12!}{9!} = 12\cdot 11\cdot 10 = 1320.$$

$$\left.\begin{array}{l} n=12 \\ k=3 \end{array}\right\} \Rightarrow \frac{12!}{(12-3)!} = P(n,k) = P(12,3)$$

Combinations

When Order Does Not Matter

$\{A, B, C\}$
$= \{A, C, B\}$
$= \{B, A, C\}$
$= \dots$

Suppose we choose a committee of size k from n people.

Now:

$$\{A, B, C\}$$

and

$$\{B, A, C\}$$

represent the same committee.

Thus order does **not** matter.

This requires a different counting formula.

Binomial Coefficients

$\{A, B, C\} \rightarrow$ ✓ABC ✓BAC ✓ACB ✓BCA

$3! = 6$

Definition

The number of subsets of size k from a set of size n is → aka the binomial coefficient

choose n , k → $$\binom{n}{k} = \boxed{\frac{n!}{k!(n-k)!}}.$$

aka $C(n,k)$

$P(n,k) = \frac{n!}{(n-k)!}$

This is read as:

"n choose k".

Note: $\binom{1}{1} = \binom{1}{0} = \binom{0}{0} = 1$ can be justified by the formula or combinatorially.

$C(n,k) = \frac{P(n,k)}{k!}$

Why Divide by $k!$?

✓

Suppose we first count ordered selections:

$$P(n, k) = \frac{n!}{(n-k)!}.$$

But each unordered subset is counted many times.

For example $\{A, B, C\}$ can appear as:

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA.$$

There are $3! = 6$ orderings.
Therefore we divide by $k!$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!} \Rightarrow \boxed{\binom{n}{k} \le P(n,k).}$$

Example: Poker Hands

A poker hand consists of 5 cards from a 52-card deck.
Order does not matter.

Thus the number of poker hands is

$$\binom{52}{5} = \frac{52!}{5!47!}.$$

at home

Example: Committees

How many committees of size 4 can be formed from 10 people?

(at home)

Since order does not matter:

$$\binom{10}{4} = \frac{10!}{4!6!}.$$

Simplifying:

$$\binom{10}{4} = 210.$$

Properties of Binomial Coefficients

Symmetry Identity

Identity

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

Reason:
Choosing k elements to include is equivalent to choosing $n-k$ elements to exclude. :)
Algebraic Proof:

Khayyam–Pascal's Identity

Khayyam-Pascal's Identity

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

Combinatorial Proof of Khayyam-Pascal's Identity

Suppose a committee of size k is selected from n people.
Focus on one distinguished person.
Two possibilities:

1. The distinguished person is not selected.
   Then choose all k members from the remaining $n-1$ people:
   $$\binom{n-1}{k}.$$
2. The distinguished person is selected.
   Then choose the remaining $k-1$ members from the remaining $n-1$ people:
   $$\binom{n-1}{k-1}.$$

Adding the two cases gives (why addition?)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Khayyam–Pascal Triangle: The Main Idea

The Khayyam–Pascal triangle is a triangular arrangement of numbers.

Each row begins and ends with 1.

Every interior entry is obtained by adding the two entries directly above it:

new entry = upper-left entry + upper-right entry

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Source: Wikipedia: Pascal's Triangle.

Building the Triangle

Start with 1 at the top.

Each new number is the sum of the two numbers above it.

1
1 1
1 $\boxed{1+1}$ 1
1 $\boxed{1+2}$ $\boxed{2+1}$ 1
1 $\boxed{1+3}$ $\boxed{3+3}$ $\boxed{3+1}$ 1

The Formula Inside the Triangle

$$
\begin{array}{lccccccccccc}
\text{row } 0 & & & & & & \binom{0}{0} & & & & & \\
\text{row } 1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
\text{row } 2 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
\text{row } 3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & & & & & & \vdots & & & & & \\
\text{row } n-1 & & \binom{n-1}{0} & \cdots & \binom{n-1}{k-1} & & \binom{n-1}{k} & & \cdots & & \binom{n-1}{n-1} & \\
\text{row } n & \binom{n}{0} & \cdots & \binom{n}{k-1} & & \binom{n}{k} & & \binom{n}{k+1} & & \cdots & & \binom{n}{n}
\end{array}
$$

Each interior entry is obtained by adding the two entries directly above it:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

This identity is known as **Khayyam-Pascal's Rule**.

Khayyam-Pascal's Rule/Identity

The rule for building the triangle is

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $1 \le k \le n-1$.

For example,

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6.$$

So every interior entry is obtained by adding the two entries above it.

Connection to Binomial Expansions

The rows of the triangle give the coefficients in binomial expansions.

$$
\begin{gathered}
(a+b)^0 = 1 \\
(a+b)^1 = a + b \\
(a+b)^2 = a^2 + 2ab + b^2 \\
(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{gathered}
$$

- The coefficients are exactly the rows of the Khayyam–Pascal triangle.
- Hence, we can show that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ (a very useful identity in Probability theory.
- Also proves why $2^n = \sum_{i=0}^{n} \binom{n}{k}$.
  Hint: $2 = 1 + 1$:)

Story Proofs and Overcounting

Story Proofs

A **story proof** proves an identity by showing that both sides count the same object in different ways.

Advantages:

- gives intuition;
- explains why identities are true;
- often simpler than algebraic proofs.

Example Story Proof

Identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Why?

The left-hand side counts subsets of an n-element set by grouping according to subset size.

The right-hand side counts subsets directly (see slide 60):
each element is either:

included or not included.

Thus there are

$$2^n$$

subsets total.

Mini-Exercises

1. How many binary strings of length 8 are there?
2. How many 3-letter strings can be formed from the English alphabet if repetition is allowed?
3. How many ordered arrangements of 4 people can be selected from 10 people?
4. How many committees of size 3 can be formed from 8 people?

Answers

$$2^8 = 256, \quad 26^3, \quad \frac{10!}{6!}, \quad \binom{8}{3} = 56.$$

Summary

- Number of ordered length-k sequences from n options (repetition allowed):
$$n^k.$$
- Number of ordered length-k sequences from n distinct objects (no repetition):
$$\frac{n!}{(n-k)!}.$$
- Number of permutations of k objects chosen from n:
$$P(n,k) = \frac{n!}{(n-k)!}.$$
- Number of combinations of k objects chosen from n:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
- Fundamental identities for binomial coefficients:
$$\binom{n}{k} = \binom{n}{n-k},$$
and
$$\binom{n}{} = \binom{n-1}{} + \binom{n-1}{}$$