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Chapter 5B: Normal and Exponential Distributions

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- define the standard Normal distribution;
- standardize a Normal random variable;
- compute Normal probabilities using the standard Normal CDF;
- use the 68–95–99.7 rule;
- define the Exponential distribution;
- compute Exponential probabilities;
- understand the memoryless property of the Exponential distribution.

Normal Distribution

The Normal Distribution $N(\mu, \sigma^2)$

mu sigma squared

Definition

A continuous random variable X has a Normal distribution with mean μ and variance σ^2 if

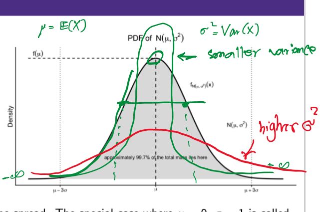
$$X \sim N(\mu, \sigma^2),$$

and its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

$0 \le$

$\int = 1$



The parameter μ controls the center, while σ^2 controls the spread. The special case where $\mu = 0$, $\sigma = 1$ is called Standard Normal.

Preview

This is the most fundamental distribution in probability, with so many important fetures. Central Limit Theorem (CLT) will later shows one of its key aspects.

PDF Verification of Normal

$f_X(x) \ge 0$ is straightforward, so it suffices to show

$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}} dx = 1$ $\quad \int \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$

Claim

$\mathbb{R} \ni \mu = 0$, $0 < \sigma^2 = 1$

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

$\int f(x) = 1$

Let

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx.$$

Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dx\, dy.$$

Using polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

$e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}} = e^{-\frac{x^2}{2} - \frac{y^2}{2}}$

Therefore,

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta.$$

Let $u = r^2/2$, so $du = r\,dr$. Then

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-u}\, du\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

Thus,

$$I = \sqrt{2\pi}.$$

Mean and Variance of $N(\mu, \sigma^2)$

Proposition

Suppose

$$X \sim N(\mu, \sigma^2),$$

with density

pdf $\rightarrow$
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Then,

$$\mathbb{E}(X) = \mu \qquad Var(X) = \sigma^2$$

$\int f(x) = 1$
$\rightarrow$ when $X \sim N(\mu = 0, \sigma^2 = 1)$
$\rightarrow \int_{-\infty}^{\infty} f(x) = 1$
$X \sim N(\mu, \sigma^2)$
$Z = \frac{X - \mu}{\sigma}$

Proof: By definition,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Let

$$z = x - \mu.$$

Then

$$x = z + \mu, \qquad dx = dz.$$

Therefore,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} (z + \mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/(2\sigma^2)}\, dz.$$

$\int f(x) = 1$
$\int x \cdot f(x) = \mu$
$\mu = 0$
$\sigma^2 = 1$

Splitting the integral,

$$\mathbb{E}(X) = \underbrace{\int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/(2\sigma^2)}\, dz}_{=0} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/(2\sigma^2)}\, dz}_{=1}.$$

Hence,

$$\mathbb{E}(X) = \mu$$

please verify at home (very important)

Variance of $N(\mu, \sigma^2)$

Recall:

$$Var(X) = \mathbb{E}[(X - \mu)^2].$$

Using LOTUS,

$$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx.$$

Let

$$z = x - \mu.$$

Then

$$Var(X) = \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/(2\sigma^2)}\, dz.$$

Now substitute

$$u = \frac{z}{\sigma}, \qquad z = \sigma u, \qquad dz = \sigma\, du.$$

Then

$$Var(X) = \sigma^2 \int_{-\infty}^{\infty} u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du.$$

The remaining integral equals 1, since it is the variance of the standard Normal distribution $N(0, 1)$. Therefore,

$$Var(X) = \sigma^2$$

Standard Normal CDF

Normal dist. $N(\mu, \sigma^2)$ $\begin{cases} \mu \in \mathbb{R} \\ \sigma^2 \in [0, \infty) \end{cases}$ $\Rightarrow$ The special case where $\mu = 0, \sigma^2 = 1$ is called standard Normal distribution

The standard Normal CDF is denoted by

$$\Phi(z).$$

It is defined by

$$\Phi(z) = \mathbb{P}(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

Important

There is no elementary closed-form expression for $\Phi(z)$.

So Normal probabilities are usually computed using tables, calculators, or software. See "normal-table.pdf" in Canvas for the Standard Normal table.

pdf General Normal $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$ | standard Normal ($\mu = 0, \sigma^2 = 1$) $\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$

Symmetry of the Standard Normal

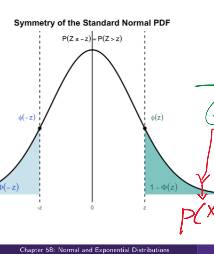
The standard Normal PDF is symmetric about 0, i.e., $\varphi(z) = \varphi(-z)$. Therefore, $\mathbb{P}(Z \le -z) = \mathbb{P}(Z \ge z)$. In terms of the CDF, $\Phi(-z) = 1 - \Phi(z)$. This is why some Normal PDF (not CDF) tables skip the negative values and only provide $\phi(z)$ for $z \ge 0$.

CDF: $F(x) = \int_{-\infty}^{x} f(u)\, du$

$X \sim N(\mu, \sigma^2)$

$P(X > 2) = 1 - P(X \le 2)$

$P(1 < X < 3) = P(X < 3) - P(X \le 1) = F(3) - F(1)$



$X \sim N(0,1)$ (std normal)

$P(X \le 1.7) \leftarrow$ check the table.

$X \sim N(3, 5^2)$

$Z = \frac{X - 3}{5} \sim N(0,1)$

$P(X < 10) = P(X - 3 < 10 - 3) = P\left(\frac{X-3}{5} < \frac{10-3}{5}\right) = P(Z \le 1.4)$ std normal $= 0.9194$ form table

Converting $N(0, 1)$ to $N(\mu, \sigma^2)$

Proposition

If

$$Z \sim N(0, 1),$$

then

$$X = \mu + \sigma Z$$

has the Normal distribution with mean μ and variance σ^2, i.e.,

$$X \sim N(\mu, \sigma^2).$$

$f: \mathbb{R} \to [0, 1]$ $\quad \sqrt{x}$

support$(f) = \{x \in Dom(f) \mid f(x) \ne 0\}$

Converting $N(\mu, \sigma^2)$ to $N(0, 1)$

Range(f) = Image(f)

Proposition

If

$$X \sim N(\mu, \sigma^2),$$

then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

$y = x^2$ $\quad$ Dom $= (-\infty, \infty)$, Range $= [0, \infty)$, supp $= (-\infty, 0) \cup (0, \infty)$

Standardization converts a general Normal random variable into a standard Normal.

Use

To compute probabilities involving X, convert them into probabilities involving Z, and then use the Normal probability table.

Example: Standardizing

Suppose

$$X \sim N(70, 10^2).$$

Find

$$\mathbb{P}(X < 85).$$

Standardize:

$$\mathbb{P}(X < 85) = \mathbb{P}\left(\frac{X - 70}{10} < \frac{85 - 70}{10}\right).$$

Thus

$$\mathbb{P}(X < 85) = \mathbb{P}(Z < 1.5) = \Phi(1.5).$$

The 68–95–99.7 Rule

If

$$X \sim N(\mu, \sigma^2),$$

then approximately:

$$\mathbb{P}(|X - \mu| < \sigma) \approx 0.68,$$

$$\mathbb{P}(|X - \mu| < 2\sigma) \approx 0.95,$$

$$\mathbb{P}(|X - \mu| < 3\sigma) \approx 0.997.$$

Interpretation

Most Normal observations lie within a few standard deviations of the mean.

Exponential Distribution

Exponential Distribution

The Exponential distribution models waiting times.

Examples:

- waiting time until a phone call;
- waiting time until a radioactive particle decays;
- waiting time until a customer arrives;
- lifetime of certain components.

It is the continuous analog of waiting for the first success.

Definition of Exponential Distribution

Definition

A random variable X has the Exponential distribution with rate $\lambda > 0$ if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0.$$

We write

$$X \sim Exp(\lambda).$$

For $x \le 0$,

$$f(x) = 0.$$

Exponential CDF and Survival Function

If

$$X \sim exp(\lambda),$$

then for $x > 0$,

$$F(x) = \mathbb{P}(X \le x) = 1 - e^{-\lambda x}.$$

Therefore,

$$\mathbb{P}(X > x) = 1 - F(x) = e^{-\lambda x}.$$

Survival function

$$\boxed{\mathbb{P}(X > x) = e^{-\lambda x}.}$$

Mean and Variance of Exponential

If

$$X \sim Exp(\lambda),$$

then

$$\mathbb{E}(X) = \frac{1}{\lambda},$$

and

$$Var(X) = \frac{1}{\lambda^2}.$$

Interpretation

A larger rate λ means events occur more frequently, so waiting times become shorter on average.

Proof of the Mean

The density of an exponential random variable is

$$f(x) = \lambda e^{-\lambda x}, \qquad x \ge 0.$$

Thus

$$\mathbb{E}(X) = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx.$$

Use integration by parts.
Let

$$u = x, \qquad dv = \lambda e^{-\lambda x}\, dx.$$

Then

$$du = dx, \qquad v = -e^{-\lambda x}.$$

Therefore,

$$\mathbb{E}(X) = \left[-x e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx.$$

The boundary term equals 0, so

$$\mathbb{E}(X) = \int_0^{\infty} e^{-\lambda x}\, dx = \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{\infty}.$$

Hence

$$\mathbb{E}(X) = \frac{1}{\lambda}.$$

Proof of the Variance

We first compute

$$\mathbb{E}(X^2).$$

Using the density,

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx.$$

Applying integration by parts twice gives

$$\mathbb{E}(X^2) = \frac{2}{\lambda^2}.$$

Now use

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Therefore,

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2.$$

Hence

$$Var(X) = \frac{1}{\lambda^2}.$$

Memoryless Property of $Exp(\lambda)$

Memoryless Property

If

$$X \sim exp(\lambda),$$

then for $s, t \ge 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Interpretation:
Given that we have already waited s units of time, the additional waiting time behaves like a fresh Exponential random variable.
Remark: Exponential distribution is the only memoryless r.v. among all the continuous r.v.s. Also, the only memoryless discrete r.v. is the Geometric r.v. that we studied in Chapter 3.

Proof of Memorylessness

Using conditional probability,

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)}.$$

For an Exponential random variable,

$$\mathbb{P}(X > x) = e^{-\lambda x}.$$

Thus,

$$\frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}.$$

But

$$e^{-\lambda t} = \mathbb{P}(X > t).$$

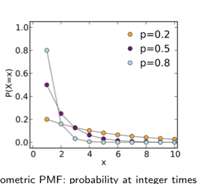
Therefore,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

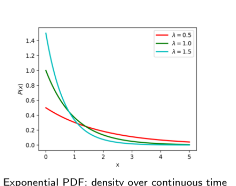
Geometric vs Exponential

Both distributions model a **waiting time until the first event**.

Distribution	Time type	Function	Formula
Geom(p)	discrete	PMF	$\mathbb{P}(X = x) = \frac{p}{1-p} e^{\ln(1-p)x}, \quad x \in \mathbb{Z}^+$
exp(λ)	continuous	PDF	$f(x) = \lambda e^{-\lambda x}, \quad x \in \mathbb{R}^+$



Geometric PMF: probability at integer times



Exponential PDF: density over continuous time

Summary

- The Normal distribution is bell-shaped and central in statistics.
- Standard Normal:
$$Z \sim N(0, 1).$$
- Standardization:
$$\frac{X - \mu}{\sigma} \sim N(0, 1).$$
- Exponential random variables model waiting times.
- If $X \sim exp(\lambda)$, then
$$\mathbb{P}(X > x) = e^{-\lambda x}.$$
- The Exponential distribution is memoryless.

Next lecture

Joint distributions and transformations.