chapter6b_mgfs

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Chapter 6B: Moment Generating Functions

MATH/STAT 394: Probability I

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previous slides

Moments: $\mu'_n = \mathbb{E}(X^n)$ are connected to MGF

this slide

Learning Goals

By the end of this lecture, students should be able to:

- define the moment generating function (MGF);
- compute MGFs of common distributions;
- extract moments from MGFs; → connection
- understand how MGFs determine distributions; → uniqueness theorem
- compute distributions of sums using MGFs;
- derive moments of Exponential and Normal distributions using MGFs.

CDF $F_{X+Y}(z) = P(X+Y \le z) = \cdots$

Definition of the MGF

Definition

The moment generating function (MGF) of X is

$$M_X(t) = \mathbb{E}(e^{tX})$$

provided the expectation is finite in some open interval around 0.

Equivalently,

$$M_X(t) = \begin{cases} \sum_x e^{tx} p_X(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx, & \text{if } X \text{ is continuous.} \end{cases}$$

$M_X(1) = \mathbb{E}(e^{1\cdot X}) = \mathbb{E}(e^X)$

$M_X(0) = \mathbb{E}(e^{0\cdot X}) = \mathbb{E}(e^0) = \mathbb{E}(1) = 1$

Quick check

Every valid MGF satisfies

$$M_X(0) = 1.$$

$X \sim f$

$M_X(t) = \mathbb{E}[e^{t\cdot X}] = \begin{cases} \sum_{t \le x} e^{t\cdot x} P_X(x) & \text{discrete} \\ \int e^{t\cdot x} f_X(x)\,dx & \text{cont.} \end{cases}$

MGFs: Discrete Distributions I

$X = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

$MGF_X(t) = M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x_i} e^{tx_i} P(X = x_i)$

$= e^{t\cdot 0} P(X=0) + e^{t\cdot 1} P(X=1)$

$= P(X=0) + e^t \cdot P(X=1)$

$= 1-p + e^t \cdot p$

$$M_X(t) = \mathbb{E}(e^{tX}) = \sum_x e^{tx} p_X(x).$$

Bernoulli

If $X \sim \text{Bern}(p)$, then

$$M_X(t) = e^{t\cdot 1} p + e^{t\cdot 0}(1-p) = pe^t + q.$$

Discrete Uniform

If $X \sim \text{DUnif}\{1, \ldots, m\}$, then

$$M_X(t) = \frac{1}{m}\sum_{k=1}^{m} e^{tk} = \frac{1}{m}(e^t + e^{2t} + \cdots + e^{mt}) = \frac{e^t(e^{mt}-1)}{m(e^t-1)}.$$

$\sum_{x_i}^{n} e^{tx_i} \cdot p(x_i) =$ $1/m$

MGFs: Discrete Distributions II

$\underset{\text{binomial}}{X} = \underbrace{X_1 + \cdots + X_n}_{\text{summation of } n \text{ bernoulli}}$ indep.

$[e^{tk}] \cdot p^k = (pe^t)^k$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ $\begin{cases} a = pe^t \\ b = q \end{cases}$

Binomial

If $X \sim \text{Bin}(n, p)$, then

$q = 1-p$

$$M_X(t) = \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k q^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(pe^t)^k q^{n-k} = (q+pe^t)^n.$$

PMF(k)

Poisson

If $X \sim \text{Pois}(\lambda)$, then

pmf of poisson

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$M_{\text{Bin}(n,p)}(t) = [M_{\text{Ber}(p)}(t)]^n$

(Theorem) $Z = X + Y$ $X \perp Y$ $\Rightarrow MGF_Z(t) = MGF_X(t) \times MGF_Y(t)$

MGFs: Discrete Distributions III

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $|x| < 1$ $x = qe^t$ $p(x) = (1-p)^x \cdot p$

Geometric

If $X \sim \text{Geom}(p)$, where X counts failures before the first success, then

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} q^k p = p\sum_{k=0}^{\infty}(qe^t)^k = \frac{p}{1-qe^t}, \quad qe^t < 1.$$

pmf

Negative Binomial

If $X \sim \text{NegBin}(r, p)$, where X counts failures before the r-th success, then

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk}\binom{k+r-1}{k}p^r q^k = p^r\sum_{k=0}^{\infty}\binom{k+r-1}{k}(qe^t)^k = \left(\frac{p}{1-qe^t}\right)^r.$$

$N.B(r,p) = \underbrace{\text{Geom}(p) + \cdots + \text{Geom}(p)}_{r \text{ times}}$

MGFs: Discrete Distributions IV

(at home)

Hypergeometric

If

$$X \sim \text{HGeom}(w, b, n),$$

then

$$\mathbb{P}(X = k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}},$$

where

$$\max(0, n-b) \le k \le \min(w, n).$$

Therefore,

$$M_X(t) = \sum_{k=\max(0,n-b)}^{\min(w,n)} e^{tk}\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}.$$

Note: Unlike Binomial or Poisson, this MGF usually does not simplify to a nice elementary closed form.

MGFs: Continuous Distributions I

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx.$$

Uniform

If $X \sim \text{Unif}(a, b)$, then

$$M_X(t) = \int_a^b e^{tx}\frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{e^{tb}-e^{ta}}{t(b-a)}, \quad t \ne 0.$$

$f_X(x)$

~~Also, $M_X(0) = 1$.~~

$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{o.w.} \end{cases}$

MGFs: Continuous Distributions II

$\int e^{-(\lambda-t)x}\,dx = \frac{-1}{\lambda-t}e^{-(\lambda-t)x}\Big|_0^{\infty}$

Exponential

If $X \sim \text{Exp}(\lambda)$, then

$$M_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{-(\lambda-t)x}\,dx = \frac{\lambda}{\lambda-t}, \quad t < \lambda.$$

Gamma

we haven't yet covered Gamma

If $X \sim \text{Gamma}(\alpha, \lambda)$, with shape α and rate λ, then

$$M_X(t) = \int_0^\infty e^{tx}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\lambda-t)x}\,dx.$$

Let $u = (\lambda-t)x$. Then $x = u/(\lambda-t)$, $dx = du/(\lambda-t)$, and

$$M_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{1}{(\lambda-t)^\alpha}\underbrace{\int_0^\infty u^{\alpha-1}e^{-u}\,du}_{\Gamma(\alpha)} = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \quad t < \lambda.$$

MGF of $N(0,1)$

$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ $x \in \mathbb{R}$

Standard Normal

Let $Z \sim N(0,1)$. Then

$$M_Z(t) = \mathbb{E}(e^{tZ}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tz}e^{-z^2/2}\,dz.$$

$e^{tz - \frac{z^2}{2}}$

Complete the square:

$$tz - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2tz) = -\frac{1}{2}((z-t)^2 - t^2) = -\frac{(z-t)^2}{2} + \frac{t^2}{2}.$$

important

Therefore,

$$M_Z(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(z-t)^2/2}e^{t^2/2}\,dz = e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(z-t)^2/2}\,dz.$$

$= \sqrt{2\pi}$

Since the last integral equals $\sqrt{2\pi}$,

$$M_Z(t) = e^{t^2/2}.$$

$f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$ $\mu = t$ $\sigma^2 = 1$

MGF of $N(\mu, \sigma^2)$

Normal

Let $X \sim N(\mu, \sigma^2)$. Write

$$X = \mu + \sigma Z, \qquad Z \sim N(0,1).$$

Then

$$M_X(t) = \mathbb{E}(e^{tX}) = \mathbb{E}\left(e^{t(\mu+\sigma Z)}\right) = e^{\mu t}\mathbb{E}(e^{\sigma tZ}).$$

Using

$$M_Z(s) = e^{s^2/2},$$

with $s = \sigma t$, we get

$$M_X(t) = e^{\mu t}M_Z(\sigma t) = e^{\mu t}e^{\sigma^2 t^2/2} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

$M_Z(t) = e^{\frac{t^2}{2}}$ $\mu = 0$ $\sigma^2 = 1$

$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Moments from MGFs

MGFs Generate Moments

The relation between pop. moments & MGFs.

Theorem

The n^{th} population moment is obtained by differentiating the MGF:

$$\mathbb{E}(X^n) = M_X^{(n)}(0).$$

pop. moment MGF

Examples:

$$\mathbb{E}(X) = M_X'(0), \qquad \mathbb{E}(X^2) = M_X''(0).$$

$\frac{d}{dt}M_X(t) = \frac{d}{dt}\mathbb{E}(e^{tX}) = \mathbb{E}\left[\frac{d}{dt}e^{tX}\right] = \mathbb{E}[X \cdot e^{t\cdot X}] \xrightarrow{t=0} M_X'(0) = \mathbb{E}[X]$

second derivative: $\mathbb{E}[X \cdot X \cdot e^{tX}] = \mathbb{E}[X^2 \cdot e^{tX}]$

Proof: MGFs Generate Moments

Recall that the moment generating function is

$$M_X(t) = \mathbb{E}(e^{tX}).$$

Differentiate once:

$$M_X'(t) = \frac{d}{dt}\mathbb{E}(e^{tX}) = \mathbb{E}\left(\frac{d}{dt}e^{tX}\right) = \mathbb{E}(Xe^{tX}).$$

Now plugging in $t = 0$,

$$M_X'(0) = \mathbb{E}(Xe^0) = \mathbb{E}(X).$$

Differentiate another time noting that X is a constant w.r.t. t.

$$M_X''(t) = \frac{d}{dt}\mathbb{E}(Xe^{tX}) = \mathbb{E}\left(\frac{d}{dt}Xe^{tX}\right) = \mathbb{E}(X^2e^{tX}).$$

Plugging in $t = 0$,

$$M_X''(0) = \mathbb{E}(X^2e^0) = \mathbb{E}(X^2).$$

Similarly, differentiating n times gives

$$M_X^{(n)}(t) = \mathbb{E}(X^ne^{tX}).$$

Therefore, at $t = 0$,

$$M_X^{(n)}(0) = \mathbb{E}(X^ne^0) = \mathbb{E}(X^n).$$

$\frac{d^n}{dt^n}M_X(t)\Big|_{t=0} = \mathbb{E}(X^n)$

Another way of seeing it...

The Taylor expansion of

$$e^{tX}$$

is

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \cdots.$$

Taking expectations:

$$M_X(t) = 1 + t\mathbb{E}(X) + \frac{t^2}{2!}\mathbb{E}(X^2) + \cdots.$$

The moments appear as Taylor coefficients.

Example: Bernoulli Moments

For

$$M_X(t) = pe^t + q,$$

we compute:

$$M_X'(t) = pe^t.$$

Thus:

$$M_X'(0) = p.$$

So

$$\mathbb{E}(X) = p.$$

MGF Uniqueness Theorem

Theorem (Uniqueness of MGF)

If two random variables have the same MGF on an interval around 0, then they have the same distribution, i.e., the moment generating function uniquely determines the distribution.

Thus MGFs characterize distributions, similarly to PDFs and CDFs.

MGFs of Sums

MGF of a Sum

Theorem

If X and Y are independent, then

$$M_{X+Y}(t) = M_X(t) \times M_Y(t).$$

More generally, if $X_1, \ldots, X_n$ are independent, then

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

This is one of the main reasons MGFs are useful. Since using the MGF Uniqueness theorem, the MGF characterize the distribution in full.

Binomial MGF

A Binomial random variable can be written as:

$$X = I_1 + \cdots + I_n,$$

where the I_j are independent Bernoulli(p) random variables.

Since each Bernoulli MGF is

$$pe^t + q,$$

we get

$$M_X(t) = (pe^t + q)^n.$$

Negative Binomial MGF

If

$$X \sim \text{NBin}(r, p),$$

then

$$X = G_1 + \cdots + G_r,$$

where the G_j are independent Geometric(p) random variables.

Thus,

$$M_X(t) = \left(\frac{p}{1-qe^t}\right)^r.$$

Sums via MGFs

Sum of Independent Poissons

If

$$X \sim \text{Pois}(\lambda), \qquad Y \sim \text{Pois}(\mu),$$

independently, then

$$M_X(t) = e^{\lambda(e^t-1)},$$

and

$$M_Y(t) = e^{\mu(e^t-1)}.$$

Therefore,

$$M_{X+Y}(t) = e^{(\lambda+\mu)(e^t-1)}.$$

Hence:

$$X + Y \sim \text{Pois}(\lambda + \mu).$$

Remark: This can be generalized to n independent Poisson r.v.s using induction.

Sum of Independent Normals

If

$$X_1 \sim N(\mu_1, \sigma_1^2), \qquad X_2 \sim N(\mu_2, \sigma_2^2),$$

independently, then

$$M_{X_1+X_2}(t) = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}.$$

Thus,

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Remark: This can be generalized to n independent Normal r.v.s using induction.

Distributions Closed Under Addition Under Independence

Distribution	Assumption	Sum Distribution
Bernoulli	$X_i \sim \text{Bern}(p)$	$\sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$
Binomial	$X_i \sim \text{Bin}(n_i, p)$	$\sum_{i=1}^{n} X_i \sim \text{Bin}\left(\sum n_i, p\right)$
Poisson	$X_i \sim \text{Pois}(\lambda_i)$	$\sum_{i=1}^{n} X_i \sim \text{Pois}\left(\sum \lambda_i\right)$
Negative Binomial	$X_i \sim \text{NegBin}(r_i, p)$	$\sum_{i=1}^{n} X_i \sim \text{NegBin}\left(\sum r_i, p\right)$
Geometric	$X_i \sim \text{Geom}(p)$	$\sum_{i=1}^{n} X_i \sim \text{NegBin}(n, p)$
Normal	$X_i \sim N(\mu_i, \sigma_i^2)$	$\sum_{i=1}^{n} X_i \sim N\left(\sum \mu_i, \sum \sigma_i^2\right)$
Gamma	$X_i \sim \text{Gamma}(\alpha_i, \lambda)$	$\sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\sum \alpha_i, \lambda\right)$
Exponential	$X_i \sim \text{Exp}(\lambda)$	$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$

Summary

- MGFs encode moments through derivatives.
- MGFs determine distributions.
- Independent sums correspond to products of MGFs.
- MGFs simplify moment calculations dramatically.
- The Normal and Exponential MGFs are especially important.

Next lecture

Joint distributions.