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Transformations in the Discrete Case

If X is discrete and

$$Y = g(X),$$

then, for every possible value y of Y,

$Y = g(X)$

$$\boxed{\mathbb{P}(Y = y) = \mathbb{P}\left(X \in g^{-1}(\{y\})\right) = \sum_{x:\, g(x)=y} \mathbb{P}(X = x).}$$

$P(Y=y) = P(g(X)=y)$

$= \mathbb{P}(X \in g^{-1}\{y\})$

In words: find all values of x that are mapped to y by g, and add their probabilities.

Example. Suppose

$$\mathbb{P}(X = -2) = 0.1, \quad \mathbb{P}(X = -1) = 0.2, \quad \mathbb{P}(X = 0) = 0.3, \quad \mathbb{P}(X = 1) = 0.25, \quad \mathbb{P}(X = 2) = 0.15,$$

and let

$$\boxed{Y = X^2.}$$

$X \in \{\pm 1, \pm 2, 0\}$

Then $Y \in \{0, 1, 4\}$, and

$P(X^2=1)$

$= P(X=1 \text{ or } X=-1)$

$$\mathbb{P}(Y = 0) = \mathbb{P}(X = 0) = 0.30,$$
$$\mathbb{P}(Y = 1) = \mathbb{P}(X \in \{-1, 1\}) = \mathbb{P}(X = -1) + \mathbb{P}(X = 1) = 0.45,$$
$$\mathbb{P}(Y = 4) = \mathbb{P}(X \in \{-2, 2\}) = \mathbb{P}(X = -2) + \mathbb{P}(X = 2) = 0.25.$$

Transformations of Continuous Random Variables

Change of Variables Theorem (Univariate Case)

Let X be a continuous random variable with PDF f_X and support S_X, and let

$$Y = g(X),$$

where g is differentiable and strictly monotone on S_X.
Then g has an inverse, and the support of Y is

$$S_Y = g(S_X).$$

For $y \in S_Y$,

$$\boxed{f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|}$$

and $f_Y(y) = 0$ for $y \notin S_Y$.

How to use it: solve for x in terms of y, find $|dx/dy|$, substitute into $f_X(x)$, and determine the support of Y.

Important

The theorem requires g to be one-to-one on the support of X. When it is not, we can break the support into monotone cases (examples below).

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$a \le b$ g: increasing

$g(a) \le g(b)$

$g(a) \ge g(b)$ g: decreasing

X: r.v. $Y = g(X)$

$F_Y(y) = \mathbb{P}(Y \le y)$

$= \mathbb{P}(g(X) \le y)$

$= \begin{cases} \mathbb{P}(X \le g^{-1}(y)) & X\text{: increasing} \\ \mathbb{P}(X \ge g^{-1}(y)) & X\text{: decreasing} \end{cases}$

$= 1 - P(X \le a)$

$= \begin{cases} F_X(g^{-1}(y)) & '' \\ 1 - F_X(g^{-1}(y)) & '' \end{cases}$

$\boxed{1 - F_X(g^{-1}(y))}$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \begin{cases} F_X(g^{-1}(y)) & " \\ 1 - F_X(g^{-1}(y)) & " \end{cases}$$

PDF

$$= \begin{cases} \frac{d}{dy} \boxed{F_X(g^{-1}(y))} & " \\ \frac{d}{dy} \left(1 - F_X(g^{-1}(y))\right) & " \end{cases}$$

chain rule

$$= \begin{cases} \boxed{f_X(g^{-1}(y)) \cdot \frac{d}{dy}\left(g^{-1}(y)\right)} & " \\ -\boxed{f_X(g^{-1}(y)) \cdot \frac{d}{dy}\left(g^{-1}(y)\right)} & " \end{cases}$$

Example 1: A Monotone Transformation

Let

$$X \sim N(0,1), \quad Y = e^X = g(X)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}$$

$Y = X^2$

Since

$$g(x) = e^x$$

is strictly increasing on $\mathbb{R}$, the change-of-variables theorem applies.

$X \in \mathbb{R}$

$Y \in (0, \infty)$

Step 1: Find the support of Y.
Since $e^x > 0$ for every x,

$$S_Y = (0, \infty).$$

Step 2: Solve $y = g(x)$ for x.

$$\log(y) = \log(e^x) \Rightarrow x = \log y. = g^{-1}(y)$$

Thus

$$g^{-1}(y) = \log y, \qquad \left|\frac{dx}{dy}\right| = \frac{1}{y}.$$

$f: A \to B$

$f^{-1}: B \to A$

Step 3: Apply the formula.
Since

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

we obtain

Dist. of e^X:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{y\sqrt{2\pi}} \exp\left\{-\frac{(\log y)^2}{2}\right\}, \qquad y > 0.}$$

For $y \le 0$, $f_Y(y) = 0$.
This is the **log-normal distribution**.

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Example 2: When the Transformation Is Not One-to-One

Let

$$X \sim N(0,1), \quad Y = X^2.$$

$$\mathbb{P}(X^2 \le y) = \mathbb{P}(X \le \sqrt{y})$$

Here

$$g(x) = x^2$$

is **not one-to-one** on $\mathbb{R}$:

$$g(-x) = g(x).$$

Therefore, we cannot define a single inverse g^{-1} on all of $\mathbb{R}$. Instead, use the **CDF method**. Since $Y = X^2 \ge 0$, the support is

$$S_Y = [0, \infty).$$

For $y > 0$,

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = \boxed{2\Phi(\sqrt{y}) - 1.} \tag{9}$$

Differentiate:

$$f_Y(y) = \frac{d}{dy}[2\Phi(\sqrt{y}) - 1] = 2\varphi(\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{\varphi(\sqrt{y})}{\sqrt{y}}, \qquad y > 0. \tag{10}$$

Therefore, using

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$P(a<X<b) = F(b) - F(a)$$

$$\{X^2 \le a\} = \{X \le \sqrt{a}\} \cap \{X \ge -\sqrt{a}\}$$

the PDF of Y is

$$Y = X^2$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2}, & y > 0, \\ 0, & y \le 0 \end{cases}$$

chi-squared distribution (d.f. = 1)

Transformations of Continuous Random Vectors

$g(X) \longrightarrow Y \in \mathbb{R}$ $g(X_1, \dots, X_n) \longmapsto Y = (Y_1, \dots, Y_n)$

Change of Variables Theorem (Multivariate Case)

Let $\mathbf{X} = (X_1, \dots, X_n)^T$ be a continuous random vector with joint PDF $f_{\mathbf{X}}$ and support $S_{\mathbf{X}} \subseteq \mathbb{R}^n$. Define $\mathbf{Y} = g(\mathbf{X})$, where $g : S_{\mathbf{X}} \to S_{\mathbf{Y}}$ is one-to-one and differentiable, with differentiable inverse $\mathbf{x} = g^{-1}(\mathbf{y})$.. Then the support of $\mathbf{Y}$ is $S_{\mathbf{Y}} = g(S_{\mathbf{X}})$, and

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \left| \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\right) \right|, \qquad \mathbf{y} \in S_{\mathbf{Y}}.$$

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|}$$

Here

Jacobian matrix

$$\frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

$$\left(\frac{\partial x}{\partial y}\right)_{ij} = \frac{\partial x_i}{\partial y_j}$$

is the **Jacobian matrix**. For $\mathbf{y} \notin S_{\mathbf{Y}}$, $f_{\mathbf{Y}}(\mathbf{y}) = 0$.

Example: A Transformation of Two Variables

Let X_1, X_2 be independent Exp(1) random variables, so

$$f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)}, \qquad x_1 > 0,\ x_2 > 0.$$

$f_{X_1}(x_1) = e^{-x_1}$ $x_1 > 0$

$f_{X_2}(x_2) = e^{-x_2}$ $x_2 > 0$

Define

$$Y_1 = X_1 + X_2, \qquad Y_2 = \frac{X_1}{X_1+X_2}.$$

Question: What is $f_{Y_1,Y_2}(y_1,y_2)$?

$f_{(Y_1,Y_2)}$

Since

$$y_2 = \frac{x_1}{\underbrace{x_1+x_2}_{y_1}} = \frac{x_1}{y_1},$$

$x_2 = y_1 - x_1$
$= y_1 - y_1 y_2$
$= y_1(1-y_2)$

we obtain

$$x_1 = y_1 y_2, \qquad x_2 = y_1(1-y_2).$$

We now find the support:
Since $x_1 > 0$ and $x_2 > 0$, we have $y_1 y_2 > 0$, $y_1(1-y_2) > 0$, which gives

$$\boxed{y_1 > 0, \qquad 0 < y_2 < 1.}$$

$(X_1, X_2) \longrightarrow (Y_1, Y_2)$

We now find the Jacobian:

$\frac{\partial x_1}{\partial y_1} = \frac{\partial}{\partial y_1} y_1 y_2$

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{pmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{pmatrix},$$

$\frac{\partial x_1}{\partial y_2}$

$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = \underline{ad - bc}$

so

$$\left|\det\left(\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right)\right| = |-y_1| = y_1.$$

$\frac{\partial x_2}{\partial y_1}$

Example: Finding the Joint PDF

$f_{\vec{X}}(\vec{g}^{-1}(\vec{y})) \cdot |\det(J)|$

By the change of variables theorem,

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2,\ y_1(1-y_2))\left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right|.$$

$= (x_1, x_2)$
$= (y_1 \cdot y_2,\ y_1(1-y_2))$

y_1

Since

$$x_1 + x_2 = y_1y_2 + y_1(1-y_2) = y_1,$$

we have

$$f_{X_1,X_2}(y_1y_2,\ y_1(1-y_2)) = e^{-y_1}$$

$$\begin{cases} f_{X_1X_2}(x_1,x_2) = e^{-(x_1+x_2)} \\ x_1 = y_1y_2 \\ x_2 = y_1(1-y_2) \end{cases}$$

Therefore,

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \boxed{y_1 e^{-y_1}}, & y_1 > 0,\quad 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(skip)

Notice that

$$f_{Y_1,Y_2}(y_1,y_2) = \underbrace{y_1 e^{-y_1}}_{f_{Y_1}(y_1)}\ \underbrace{1}_{f_{Y_2}(y_2)} = (y_2)$$

$\lambda e^{-\lambda x}$

Hence,

$$Y_1 \sim \mathrm{Gamma}(2,1), \qquad Y_2 \sim \mathrm{Unif}(0,1), \qquad Y_1 \perp Y_2.$$

$$f_{Unif(0,1)}(x) = \begin{cases} 1 & 0<x<1 \\ 0 & o.w. \end{cases}$$

$Z = X + Y$

when $X \perp Y$

Convolutions

What Is a Convolution?

A convolution is the distribution of a sum of independent random variables.
If

$$T = X + Y,$$

and X, Y are independent, we want the distribution of T.

Why sums matter

Sums appear everywhere:

- totals;
- averages;
- waiting times;
- counting processes.

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Discrete Convolution Formula

Theorem

If X and Y are independent discrete random variables and

$$T = X + Y,$$

(skip)

then

$$\boxed{\mathbb{P}(T = t) = \sum_{x} \mathbb{P}(X = x)\mathbb{P}(Y = t - x).}$$

Proof.

For a fixed value t, the event

$$\{X + Y = t\}$$

can be written as the disjoint union

$$\{X + Y = t\} = \bigcup_{x} \{X = x,\ Y = t - x\}.$$

Therefore,

$$\mathbb{P}(T = t) = \sum_{x} \mathbb{P}(X = x,\ Y = t - x).$$

Continuous Convolution Formula

Continuous Convolution Formula

Theorem

If X and Y are independent continuous random variables and

$$\boxed{T = X + Y,}$$

then

$$\boxed{f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t - x)\, dx.}$$

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Proof of the Continuous Convolution Formula

Since

$$T = X + Y,$$

the CDF of T is

$$F_T(t) = \mathbb{P}(T \le t) = \boxed{\mathbb{P}(X + Y \le t).}$$

Using the joint density,

$$F_T(t) = \boxed{\iint_{\{(x,y):\, x+y\le t\}} f_{X,Y}(x,y)\, dx\, dy.}$$

For each fixed x, the condition

$$x + y \le t$$

is equivalent to

$$y \le t - x.$$

Therefore,

$$F_T(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_{X,Y}(x,y)\, dy\, dx.$$

Since X and Y are independent,

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

so

$$F_T(t) = \int_{-\infty}^{\infty} f_X(x) \boxed{\left[\int_{-\infty}^{t-x} f_Y(y)\, dy\right]} dx.$$

Differentiating with respect to t,

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) \boxed{f_Y(t - x)}\, dx.$$

$P(T \in A = (-\infty, t])$

$= \iint ----$

$f_X(x) f_Y(y)$

$= F_Y(t-x)$

$= \int_{-\infty}^{\infty} f_X(x) \boxed{F_Y(t-x)} dx$

differentiate w.r.t t

$MGF_{X+Y}(t) = MGF_X(t) \cdot MGF_Y(t)$

Proof of Convolution Using the Jacobian Formula

Let
$$T = X + Y, \qquad U = X.$$
Then the transformation is
$$t = x + y, \qquad u = x.$$
Solving for the original variables gives
$$\boxed{x = u, \qquad y = t - u.}$$
The Jacobian of the inverse transformation is
$$\frac{\partial(x,y)}{\partial(t,u)} = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$
and therefore
$$\left|\det\left(\frac{\partial(x,y)}{\partial(t,u)}\right)\right| = 1.$$
By the change-of-variables formula,
$$f_{T,U}(t,u) = f_{X,Y}(u, t-u).$$
Since X and Y are independent,
$$f_{X,Y}(u, t-u) = f_X(u) f_Y(t-u).$$
Hence,
$$\boxed{f_{T,U}(t,u) = f_X(u) f_Y(t-u).}$$

(optional)

$X = (X_1, \dots, X_n)$

$Y = (Y_1, \dots, Y_n)$

$(X, Y) \to (X+Y, X)$

J

F

$X+Y, X$

Z

Proof of Convolution Using the Jacobian Formula (continued)

Proof of Convolution Using the Jacobian Formula (continued)

We have found the joint PDF

$$f_{T,U}(t,u) = f_X(u) f_Y(t-u).$$

To obtain the marginal PDF of

$$T = X + Y,$$

integrate out U:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{T,U}(t,u)\,du \\ &= \int_{-\infty}^{\infty} f_X(u) f_Y(t-u)\,du. \end{aligned}$$

Therefore,

$$f_T(t) = \int_{-\infty}^{\infty} f_X(u) f_Y(t-u)\,du.$$

Renaming the dummy variable u as x gives the previous convolution formula:

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx.$$

Example 1: Sum of Two Exponential Random Variables

Let

$$X, Y \overset{\text{i.i.d}}{\sim} \text{Exp}(\lambda), \qquad T = X + Y.$$

Recall that

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$f_{X+Y} = ?$

Since $X, Y > 0$, we immediately know

$$T = X + Y > 0.$$

For independent continuous random variables, the convolution formula is

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t - x)\, dx.$$ ← previous Thm

For the integrand to be nonzero, we need *both*

$$x > 0 \qquad \text{and} \qquad t - x > 0.$$

$= t \cdot \lambda^2 \cdot e^{-\lambda t} \ (t > 0)$

Thus,

$$0 < x < t,$$

which is possible only when $t > 0$.
Therefore, for $t > 0$,

$$f_T(t) = \int_0^t f_X(x) f_Y(t - x)\, dx$$

$$= \int_0^t \lambda e^{-\cancel{\lambda x}} \lambda e^{-\lambda(t - \cancel{x})}\, dx = \lambda^2 \cdot e^{-\lambda t} \int_0^t 1\, dx = \cdots$$

$$= \lambda^2 \int_0^t$$