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Monday, July 20, 2026 11:57 AM

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University of Washington
Department of Mathematics

Midterm Examination (Practice)

MATH 394: Probability I

Summer 2026 — Instructor: Arman Jahangiri

Date: July 22, 2026
Answering time: 60 minutes
Total points: 60

Name: ____________________
Student ID: ____________________
UW NetID: ____________________
Signature: ____________________

Instructions
- This examination contains **five problems**. Verify that your copy is complete.
- Show all work and justify your answers. Unsupported answers may receive little or no credit.
- Clearly define any events or random variables that you introduce.
- For any perceived ambiguity, state a reasonable interpretation and proceed.
- Do not remove pages from the examination booklet.

Permitted materials
Examinations are closed-book and closed-notes. Students may use:
- one handwritten 8.5 × 11 inch reference sheet (front and back), and
- a non-programmable non-graphing calculator capable of standard exponential and logarithmic computations (e.g. TI-30X IIS).

No other electronic devices, calculators, communication devices, or external resources may be used during examinations.

Problem	Score
1	/10
2	/10
3	/10
4	/15
5	/15
Total	**/60**

By signing above, you affirm that the work is your own and that you followed the examination rules.

Problem 1. **10 points**

A university student ID consists of three uppercase English letters followed by four digits. Letters and digits may repeat. The first letter cannot be X, and at least one of the four digits must be even.

How many such student IDs are possible? Justify your counting.

can't be X — at least one even digit

Task I: 26; $25 \cdot 26 \cdot 26$

Task II: 10; Total number of strings of size 4 − all odd: $10^4 - 5^4$

$\Longrightarrow |\text{Task I}| \times |\text{Task II}|$

$25 \cdot 26^2 \cdot [10^4 - 5^4]$

Problem 2. **10 points**

A warehouse receives items from three factories. Factory A supplies 40% of the items and has a defect rate of 2%. Factory B supplies 35% of the items and has a defect rate of 5%. Factory C supplies 25% of the items and has a defect rate of 8%.

An item is selected uniformly at random from the warehouse and is found to be defective.

(a) Find the probability that the item came from Factory C. **(6 points)**

(b) Find the probability that the item did not come from Factory A. **(4 points)**

D = event that item is defective $\mathbb{P}(\ \mid D)$

~~$\mathbb{P}(C)$~~ $\mathbb{P}(C \mid D)$ ✓

$P(A) = .4 \quad P(B) = .35 \quad P(C) = .25$

$P(D \mid A) = .02, \quad P(D \mid B) = .05 \quad P(D \mid C) = .08$

(a) $P(C \mid D) = \dfrac{P(D \mid C) \cdot P(C)}{P(D)} = \dfrac{(0.08)(0.25)}{0.0455} = \dfrac{40}{91}$ (Bayes rule)

(b) $\mathbb{P}(A^c \mid D) = 1 - \mathbb{P}(A \mid D) = 1 - \dfrac{P(D \mid A) P(A)}{P(D)} = 1 - \dfrac{(0.008)}{0.045} = \dfrac{75}{91}$ (Bayes's rule)

Problem 3. **10 points**

Let A, B, and C be events.

(a) Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

(6 points)

(b) Deduce that

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C).$$

(2 points)

(c) Give a simple necessary and sufficient condition for equality in part (b). **(2 points)**

(a) $P((A \cup B) \cup C) = P(D) + P(C) - P(D \cap C)$, with $D = A \cup B$

$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$ (*)

$= P(A) + P(B) - P(A \cap B) + P(C) - P[\underbrace{A \cap C}_{E} \cup \underbrace{B \cap C}_{F}]$

$= \ldots - [P(E) + P(F) - P(E \cap F)]$, where $P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

(b) From (*) in (a), we have:

$P(A \cup B \cup C) = P(A) + P(B) + P(C) \underbrace{- P(A \cap B) - P((A \cup B) \cap C)}_{\leq 0} \leq P(A) + P(B) + P(C)$

(c) ⟹ if $P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$, according to (a).

$\Rightarrow P(A \cap B \cap C) = 0 \quad (A \cap B \cap C \subseteq A \cap B) \Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C)$

⟸ if $P(A \cup B \cup C) = P(A) + P(B) + P(C)$, again, according to A, we must have: $P(A \cap B \cap C) = P(A \cap B) + P(A \cap C) + P(B \cap C)$

(since $A \cap B \cap C \subseteq A \cap B, A \cap C, B \cap C$) $P(A \cap B), P(B \cap C), P(A \cap C)$

$\Longrightarrow P(A \cap B) + P(A \cap C) + P(B \cap C) \leq P(A \cap B), P(B \cap C), P(A \cap C)$ (by *)

(1) ⟹ $P(A \cap C) + P(B \cap C) = 0 \Rightarrow P(A \cap C) = P(B \cap C) = 0$ (I) (cancel $P(A \cap B)$; each ≥ 0)

(3) ⟹ $P(A \cap B) + P(A \cap C) + P(B \cap C) \leq P(A \cap C)$

⟹ $P(A \cap B) + P(B \cap C) = 0$

⟹ $P(A \cap B) = P(B \cap C) = 0$ (II)

(I), (II) ⟹ ∎

Problem 4. **15 points**

The cumulative distribution function of a continuous random variable X is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \dfrac{x^2}{4}, & 0 \leq x < 1, \\ \dfrac{2x - 1}{4}, & 1 \leq x < 2, \\ 1 - \dfrac{(3 - x)^2}{4}, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

(a) Find the probability density function $f_X(x)$. **(4 points)**

(b) Prove that $f_X(x)$ found in part (a) is a PDF. **(3 points)**

(c) Find $P(\frac{1}{2} < X \leq \frac{5}{2})$. **(3 points)**

(d) Find $\mathbb{E}[X]$. **(5 points)**

(a) $f_X(x) = \dfrac{d}{dx} F_X(x)$

if $x < 0$, $f_X(x) = 0$

$0 < x < 1$, $f_X(x) = x/2$

$1 < x < 2$, $f_X(x) = 1/2$

$2 < x < 3$, $f_X(x) = \dfrac{3-x}{2}$

$x > 0$, $f_X(x) = 0$

$$f_X(x) = \begin{cases} x/2 & 0 < x < 1 \\ 1/2 & 1 < x < 2 \\ \frac{3-x}{2} & 2 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

Indicator function representation:

$f_X(x) = \frac{x}{2} \cdot 1\{0 < x < 1\} + \frac{1}{2} \cdot 1\{1 < x < 2\} + \frac{3-x}{2} \cdot 1\{2 < x < 3\}$

(b) $f_X(x) \geq 0$: ($f_X(x) = \frac{x}{2} \geq 0\ \forall x \in (0,1)$), ($f_X(x) = \frac{1}{2} > 0\ \forall x$), ($f_X(x) = \frac{3-x}{2} \geq 0\ \forall x \in (2,3)$), ($f_X(x) = 0\ \forall x \notin (0,1) \cup (1,2) \cup (2,3)$) ⟹ ($f_X(x) \geq 0\ \forall x \in \mathbb{R}$)

$\int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \frac{3-x}{2}\,dx = 1$

Do the integration....

(4)(c) $F(a < X < b) = F(b) - F(a)$

$\mathbb{P}(\frac{1}{2} \leq X < \frac{5}{2}) = F_X(\frac{5}{2}) - F_X(\frac{1}{2})$

$= 1 - \dfrac{(3 - \frac{5}{2})^2}{4} - \dfrac{(\frac{1}{2})^2}{4}$

$= \ldots$

(4)(d) $\mathbb{E}[X] = \int_{\mathbb{R}} x \cdot f_X(x)\,dx$

$= \int_0^1 x \left(\frac{x}{2}\right) dx + \int_1^2 x \cdot \frac{1}{2}\,dx + \int_2^3 x \left(\frac{3-x}{2}\right) dx$

$= \frac{x^3}{6}\Big]_0^1 + \frac{x^2}{4}\Big]_1^2 + \left(\frac{3x^2}{4} - \frac{x^3}{6}\right)\Big]_2^3$

$= \ldots$

$\int x \cdot e^{-x}\,dx$ I.B.P

Conditions of poisson approximation: $n \to \infty$, $p \to 0$, $\lambda = n \cdot p \to c \neq 0$

(rare events ⟹ poisson approx) ⟹ $X \sim Bin(n = 8000, p = 0.0005)$

$P(Y < a) \quad P(Y = a) \quad P(Y > a)$

(a) $X \sim Bin(8000, 0.0005)$

$\lambda = n \cdot p = 4 \neq 0$ n: large p: small.

$\Longrightarrow X \approx W \quad W \sim Poisson(\lambda = 4)$

(b) $P(X$: # of mechanical failures $\geq 3) = 1 - P(X < 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[e^{-4} + e^{-4} \cdot 4 + \dfrac{e^{-4} \cdot 4^2}{2!}\right]$

$= \ldots$

$p_W(x) = \dfrac{e^{-\lambda} \lambda^x}{x!} = \dfrac{e^{-4} \cdot 4^x}{x!}$ $(x = 0, 1, 2, 3, \ldots)$

(c) $n \cdot p > 10 \quad np(1-p) > 10 \quad n = 300,\ p = 0.35$

$np = 105 > 10$ ✓ $\quad \underbrace{n(1-p)}_{195} > 10$ ✓

$\mu = np = 105 \quad \sigma^2 = np(1-p) = 68.25$

claim: $Y \approx V \quad V \sim N(105, 68.25)$ ∎

$P(95 < Y < 115) \approx \mathbb{P}(94.5 < V < 115.5)$ (continuity correction)

$= \mathbb{P}\left(\dfrac{94.5 - 105}{\sqrt{68.25}} < Z < \dfrac{115.5 - 105}{\sqrt{68.25}}\right)$, $Z \sim N(0,1)$; -1.27, 1.27

$= \Phi(1.27) - \Phi(-1.27)$