Question Bank with Solutions (Final)

MATH 394: Probability I

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Problem 1

Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the general k-th raw moment

$$\mathbb{E}(X^k), \qquad k = 1, 2, \ldots$$

(b) Let $M_X(t)$ denote the moment generating function of X. Using your result from part (a), find

$$M_X^{(k)}(0), \qquad k = 1, 2, \ldots,$$

where $M_X^{(k)}$ denotes the k-th derivative of M_X.
In particular, find

$$M'_X(0), \qquad M''_X(0), \qquad M_X^{(3)}(0), \qquad M_X^{(4)}(0).$$

(c) Find

$$\text{Var}(X).$$

(d) Find the median m of X.

(e) Find the skewness

$$\gamma_1 = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3},$$

where

$$\mu = \mathbb{E}(X), \qquad \sigma^2 = \text{Var}(X).$$

(f) Find the kurtosis

$$\gamma_2 = \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4}.$$

Solution

(a) For $k \geq 1$,

$$\begin{aligned}
\mathbb{E}(X^k) &= \int_{-\infty}^{\infty} x^k f_X(x)\, dx \\
&= \int_0^1 x^k (2x)\, dx \\
&= 2\int_0^1 x^{k+1}\, dx \\
&= 2\left[\frac{x^{k+2}}{k+2}\right]_0^1 \\
&= \frac{2}{k+2}.
\end{aligned}$$

Therefore,

$$\mathbb{E}(X^k) = \frac{2}{k+2}.$$

(b) Recall that, whenever the moment generating function exists in a neighborhood of 0,

$$M_X^{(k)}(0) = \mathbb{E}(X^k).$$

From part (a),

$$\mathbb{E}(X^k) = \frac{2}{k+2}.$$

Therefore,

$$M_X^{(k)}(0) = \frac{2}{k+2}, \qquad k = 1, 2, \ldots$$

In particular,

$$M_X'(0) = \frac{2}{3}, \qquad M_X''(0) = \frac{1}{2}, \qquad M_X^{(3)}(0) = \frac{2}{5}, \qquad M_X^{(4)}(0) = \frac{1}{3}.$$

(c) We have

$$\mathbb{E}(X) = M_X'(0) = \frac{2}{3}$$

and

$$\mathbb{E}(X^2) = M_X''(0) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18}. \end{aligned}$$

Thus,

$$\sigma^2 = \frac{1}{18}, \qquad \sigma = \frac{1}{3\sqrt{2}}.$$

(d) First calculate the CDF. For $0 \le x \le 1$,

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= \int_0^x 2t\,dt \\ &= x^2. \end{aligned}$$

The median m satisfies

$$F_X(m) = \frac{1}{2}.$$

Hence,

$$m^2 = \frac{1}{2}.$$

Since $m \in (0,1)$,

$$m = \frac{1}{\sqrt{2}}.$$

(e) The third central moment is

$$\mathbb{E}[(X-\mu)^3].$$

Expanding,

$$(X-\mu)^3 = X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3.$$

Therefore,

$$\mathbb{E}[(X-\mu)^3] = \mathbb{E}(X^3) - 3\mu\mathbb{E}(X^2) + 3\mu^2\mathbb{E}(X) - \mu^3.$$

Using

$$\mu = \frac{2}{3}, \qquad \mathbb{E}(X^2) = \frac{1}{2}, \qquad \mathbb{E}(X^3) = \frac{2}{5},$$

we obtain

$$\begin{aligned}\mathbb{E}[(X-\mu)^3] &= \frac{2}{5} - 3\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + 3\left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^3 \\ &= \frac{2}{5} - 1 + \frac{8}{9} - \frac{8}{27} \\ &= -\frac{1}{135}.\end{aligned}$$

Also,

$$\sigma^3 = \left(\frac{1}{18}\right)^{3/2} = \frac{1}{54\sqrt{2}}.$$

Hence,

$$\begin{aligned}\gamma_1 &= \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} \\ &= \frac{-1/135}{1/(54\sqrt{2})} \\ &= -\frac{2\sqrt{2}}{5}.\end{aligned}$$

Therefore,

$$\gamma_1 = -\frac{2\sqrt{2}}{5}.$$

Since the skewness is negative, the distribution is negatively skewed.

(f) The fourth central moment is

$$\mathbb{E}[(X-\mu)^4].$$

Expanding,

$$(X-\mu)^4 = X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4.$$

Therefore,

$$\mathbb{E}[(X-\mu)^4] = \mathbb{E}(X^4) - 4\mu\mathbb{E}(X^3) + 6\mu^2\mathbb{E}(X^2) - 4\mu^3\mathbb{E}(X) + \mu^4.$$

Using

$$\mu = \frac{2}{3}, \qquad \mathbb{E}(X^2) = \frac{1}{2}, \qquad \mathbb{E}(X^3) = \frac{2}{5}, \qquad \mathbb{E}(X^4) = \frac{1}{3},$$

we obtain

$$\begin{aligned}\mathbb{E}[(X-\mu)^4] &= \frac{1}{3} - 4\left(\frac{2}{3}\right)\left(\frac{2}{5}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) \\ &\quad - 4\left(\frac{2}{3}\right)^3\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^4 \\ &= \frac{1}{3} - \frac{16}{15} + \frac{4}{3} - \frac{64}{81} + \frac{16}{81} \\ &= \frac{1}{135}.\end{aligned}$$

Also,

$$\sigma^4 = \left(\frac{1}{18}\right)^2 = \frac{1}{324}.$$

Hence,

$$\begin{aligned}\gamma_2 &= \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4} \\ &= \frac{1/135}{1/324} \\ &= \frac{12}{5}.\end{aligned}$$

Therefore,

$$\gamma_2 = \frac{12}{5}.$$

Problem 2.

Let X and Y be independent random variables with moment generating functions

$$M_X(t) = e^{2(e^t-1)}$$

and

$$M_Y(t) = \frac{3}{3-t}, \qquad t < 3.$$

(a) Find the moment generating function of

$$X + Y.$$

(b) More generally, let $a, b \in \mathbb{R}$. Find the moment generating function of

$$Z = aX + bY.$$

State the values of t for which your answer is valid.

Solution

(a) Since X and Y are independent,

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}\left(e^{t(X+Y)}\right) \\ &= \mathbb{E}\left(e^{tX}e^{tY}\right) \\ &= \mathbb{E}\left(e^{tX}\right)\mathbb{E}\left(e^{tY}\right) \\ &= M_X(t)M_Y(t). \end{aligned}$$

Therefore,

$$M_{X+Y}(t) = e^{2(e^t-1)}\frac{3}{3-t}, \qquad t < 3.$$

(b) Let

$$Z = aX + bY.$$

By definition,

$$\begin{aligned} M_Z(t) &= \mathbb{E}\left(e^{tZ}\right) \\ &= \mathbb{E}\left(e^{t(aX+bY)}\right) \\ &= \mathbb{E}\left(e^{atX}e^{btY}\right). \end{aligned}$$

Since X and Y are independent,

$$\begin{aligned} M_Z(t) &= \mathbb{E}\left(e^{atX}\right)\mathbb{E}\left(e^{btY}\right) \\ &= M_X(at)M_Y(bt). \end{aligned}$$

Now,

$$M_X(at) = e^{2(e^{at}-1)}$$

and

$$M_Y(bt) = \frac{3}{3-bt}.$$

Hence,

$$M_{aX+bY}(t) = e^{2(e^{at}-1)} \frac{3}{3-bt}.$$

The restriction comes from $M_Y(bt)$, which requires

$$bt < 3.$$

Thus,

$$M_{aX+bY}(t) = \frac{3e^{2(e^{at}-1)}}{3-bt}, \qquad bt < 3.$$

Equivalently, the domain is

$$\begin{cases} t < 3/b, & b > 0, \\ t > 3/b, & b < 0, \\ t \in \mathbb{R}, & b = 0. \end{cases}$$

Problem 3

Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} c(x+y), & x = 1,2,3, \quad y = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

Determine

(a) the value of the constant c;

(b) the marginal probability mass functions of X and Y;

(c) $P(X \geq 2 \mid Y = 1)$;

(d) $E(X)$ and $E(Y)$.

Solution

(a) Since the total probability must equal 1,

$$\sum_{x=1}^{3} \sum_{y=1}^{2} c(x+y) = 1.$$

Thus,

$$c[(2+3)+(3+4)+(4+5)] = 1.$$

Therefore,

$$21c = 1,$$

and hence

$$c = \frac{1}{21}.$$

(b) For $x = 1,2,3$,

$$\begin{aligned} p_X(x) &= \sum_{y=1}^{2} p(x,y) \\ &= \frac{1}{21}\left[(x+1)+(x+2)\right] \\ &= \frac{2x+3}{21}. \end{aligned}$$

Thus,

$$p_X(x) = \begin{cases} \dfrac{2x+3}{21}, & x = 1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

For $y = 1,2$,

$$\begin{aligned} p_Y(y) &= \sum_{x=1}^{3} p(x,y) \\ &= \frac{1}{21}\left[(1+y)+(2+y)+(3+y)\right] \\ &= \frac{6+3y}{21}. \end{aligned}$$

Therefore,

$$p_Y(y) = \begin{cases} \frac{3}{7}, & y = 1, \\ \frac{4}{7}, & y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$P(X \geq 2 \mid Y = 1) = \frac{P(X \geq 2, Y = 1)}{P(Y = 1)}.$$

The numerator is

$$\begin{aligned} P(X \geq 2, Y = 1) &= p(2,1) + p(3,1) \\ &= \frac{3}{21} + \frac{4}{21} \\ &= \frac{7}{21}. \end{aligned}$$

Also,

$$P(Y = 1) = \frac{3}{7} = \frac{9}{21}.$$

Therefore,

$$P(X \geq 2 \mid Y = 1) = \frac{7/21}{9/21} = \frac{7}{9}.$$

(d)

$$\begin{aligned} E(X) &= \sum_{x=1}^{3} x\, p_X(x) \\ &= 1\left(\frac{5}{21}\right) + 2\left(\frac{7}{21}\right) + 3\left(\frac{9}{21}\right) \\ &= \frac{46}{21}. \end{aligned}$$

Similarly,

$$E(Y) = 1\left(\frac{3}{7}\right) + 2\left(\frac{4}{7}\right) = \frac{11}{7}.$$

Thus,

$$E(X) = \frac{46}{21}, \qquad E(Y) = \frac{11}{7}.$$

Problem 4

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability density functions of X and Y.

(b) Calculate $E(X)$ and $E(Y)$.

Solution

The support of the joint density is

$$0 \le y \le x \le 1.$$

(a) For a fixed $x \in [0,1]$, the possible values of y satisfy

$$0 \le y \le x.$$

Therefore,

$$\begin{aligned} f_X(x) &= \int_0^x 8xy\,dy \\ &= 8x\left[\frac{y^2}{2}\right]_0^x \\ &= 4x^3. \end{aligned}$$

Hence,

$$f_X(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a fixed $y \in [0,1]$, the support requires

$$y \le x \le 1.$$

Thus,

$$\begin{aligned} f_Y(y) &= \int_y^1 8xy\,dx \\ &= 8y\left[\frac{x^2}{2}\right]_y^1 \\ &= 4y(1-y^2). \end{aligned}$$

Therefore,

$$f_Y(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Using the marginal density of X,

$$\begin{aligned} E(X) &= \int_0^1 x f_X(x)\,dx \\ &= \int_0^1 4x^4\,dx \\ &= \frac{4}{5}. \end{aligned}$$

Similarly,

$$\begin{aligned} E(Y) &= \int_0^1 y f_Y(y)\,dy \\ &= 4\int_0^1 y^2(1-y^2)\,dy \\ &= 4\left(\frac{1}{3}-\frac{1}{5}\right) \\ &= \frac{8}{15}. \end{aligned}$$

Thus,

$$E(X) = \frac{4}{5}, \qquad E(Y) = \frac{8}{15}.$$

Problem 5

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} \frac{1}{2}ye^{-x}, & x > 0, \quad 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the marginal probability density functions of X and Y.

Solution

For $x > 0$,

$$\begin{aligned} f_X(x) &= \int_0^2 \frac{1}{2}ye^{-x}\,dy \\ &= \frac{1}{2}e^{-x}\left[\frac{y^2}{2}\right]_0^2 \\ &= e^{-x}. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < y < 2$,

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{1}{2}ye^{-x}\,dx \\ &= \frac{y}{2}\int_0^\infty e^{-x}\,dx \\ &= \frac{y}{2}. \end{aligned}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{y}{2}, & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6

Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} 1, & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate

$$P\left(X + Y \le \frac{1}{2}\right), \qquad P\left(X - Y \le \frac{1}{2}\right),$$
$$P\left(XY \le \frac{1}{4}\right), \qquad P(X^2 + Y^2 \le 1).$$

Solution

Since the density is equal to 1 on the unit square, each desired probability is equal to the area of the corresponding region inside $[0,1]^2$.
First,

$$X + Y \le \frac{1}{2}$$

describes a right triangle with legs of length $1/2$. Hence,

$$P\left(X + Y \le \frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}.$$

Next,

$$X - Y \le \frac{1}{2}.$$

It is easier to subtract the region

$$X - Y > \frac{1}{2}.$$

This is a right triangle in the lower-right portion of the unit square with legs of length $1/2$. Therefore,

$$P\left(X - Y > \frac{1}{2}\right) = \frac{1}{8},$$

and hence

$$P\left(X - Y \le \frac{1}{2}\right) = 1 - \frac{1}{8} = \frac{7}{8}.$$

For the third probability,

$$XY \le \frac{1}{4}.$$

If $0 \le x \le 1/4$, every $0 \le y \le 1$ satisfies the inequality.
If $1/4 < x \le 1$, then

$$y \le \frac{1}{4x}.$$

Thus,

$$\begin{aligned} P\left(XY \le \frac{1}{4}\right) &= \int_0^{1/4}\int_0^1 dy\,dx + \int_{1/4}^1\int_0^{1/(4x)} dy\,dx \\ &= \frac{1}{4} + \frac{1}{4}\int_{1/4}^1 \frac{1}{x}\,dx \\ &= \frac{1}{4} + \frac{1}{4}\ln 4. \end{aligned}$$

Finally,

$$X^2 + Y^2 \leq 1$$

describes the quarter of the unit disk lying in the first quadrant. Therefore,

$$P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}.$$

Hence,

$$P\left(X + Y \leq \frac{1}{2}\right) = \frac{1}{8},$$
$$P\left(X - Y \leq \frac{1}{2}\right) = \frac{7}{8},$$
$$P\left(XY \leq \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4}\ln 4,$$

and

$$P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}.$$

Problem 7

Suppose that h is the probability density function of a continuous random variable. Let the joint probability density function of X and Y be given by

$$f(x, y) = h(x)h(y), \qquad x, y \in \mathbb{R}.$$

Prove that

$$P(X \geq Y) = \frac{1}{2}.$$

Solution

Since

$$f(x, y) = h(x)h(y),$$

X and Y are independent and identically distributed continuous random variables.
Because X and Y have the same distribution, symmetry gives

$$P(X > Y) = P(Y > X).$$

Moreover, since X and Y are continuous,

$$P(X = Y) = 0.$$

The events

$$\{X > Y\}, \qquad \{Y > X\}, \qquad \{X = Y\}$$

partition the sample space. Therefore,

$$P(X > Y) + P(Y > X) + P(X = Y) = 1.$$

Hence,

$$2P(X > Y) = 1,$$

so

$$P(X > Y) = \frac{1}{2}.$$

Finally,

$$P(X \geq Y) = P(X > Y) + P(X = Y) = \frac{1}{2}.$$

Problem 8

Let X and Y be continuous random variables with joint probability density function $f(x, y)$. Define

$$Z = \frac{Y}{X}, \qquad X \neq 0.$$

Prove that the probability density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f(x, xz)\, dx.$$

Solution

Introduce the transformation

$$Z = \frac{Y}{X}, \qquad W = X.$$

Then

$$z = \frac{y}{x}, \qquad w = x.$$

Solving for x and y in terms of z and w,

$$x = w, \qquad y = wz.$$

The Jacobian matrix of the inverse transformation is

$$\frac{\partial(x, y)}{\partial(z, w)} = \begin{pmatrix} 0 & 1 \\ w & z \end{pmatrix}.$$

Therefore,

$$\det \frac{\partial(x, y)}{\partial(z, w)} = -w,$$

and hence

$$\left| \det \frac{\partial(x, y)}{\partial(z, w)} \right| = |w|.$$

By the change-of-variables formula,

$$f_{Z,W}(z, w) = f(w, wz)|w|.$$

To obtain the marginal density of Z, integrate out W:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Z,W}(z, w)\, dw \\ &= \int_{-\infty}^{\infty} |w| f(w, wz)\, dw. \end{aligned}$$

Renaming the variable of integration from w to x,

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f(x, xz)\, dx.$$

Problem 9

Prove that random variables X and Y with joint probability density function

$$f(x,y) = \begin{cases} 8xy, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

are not independent.

Solution

We first calculate the marginal densities.
For $0 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_x^1 8xy\,dy \\ &= 8x\left[\frac{y^2}{2}\right]_x^1 \\ &= 4x(1-x^2). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 4x(1-x^2), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \le y \le 1$,

$$\begin{aligned} f_Y(y) &= \int_0^y 8xy\,dx \\ &= 8y\left[\frac{x^2}{2}\right]_0^y \\ &= 4y^3. \end{aligned}$$

Hence,

$$f_Y(y) = \begin{cases} 4y^3, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

If X and Y were independent, we would have

$$f(x,y) = f_X(x)f_Y(y)$$

for all x, y.
However,

$$f_X(x)f_Y(y) = 16xy^3(1-x^2),$$

which is not equal to

$$f(x,y) = 8xy$$

on the support.
Therefore, X and Y are not independent.
There is also a geometric reason for the dependence. The support

$$0 \le x \le y \le 1$$

is triangular rather than a Cartesian product of one-dimensional supports. In particular, knowing the value of Y restricts the possible values of X.

Problem 10

Let the joint probability mass function of random variables X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{7}x^2 y, & (x,y) = (1,1),(1,2),(2,1), \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent? Why or why not?

Solution

The nonzero joint probabilities are

$$p(1,1) = \frac{1}{7}, \qquad p(1,2) = \frac{2}{7}, \qquad p(2,1) = \frac{4}{7}.$$

Therefore,

$$P(X=1) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7},$$

and

$$P(X=2) = \frac{4}{7}.$$

Similarly,

$$P(Y=1) = \frac{1}{7} + \frac{4}{7} = \frac{5}{7},$$

and

$$P(Y=2) = \frac{2}{7}.$$

If X and Y were independent, then

$$P(X=2, Y=2) = P(X=2)P(Y=2).$$

However,

$$P(X=2, Y=2) = 0,$$

whereas

$$P(X=2)P(Y=2) = \frac{4}{7}\frac{2}{7} = \frac{8}{49} > 0.$$

Therefore,

$$P(X=2, Y=2) \neq P(X=2)P(Y=2),$$

so X and Y are not independent.

Problem 11

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} x^2 e^{-x(y+1)}, & x \geq 0, \quad y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent? Why or why not?

Solution

We calculate the marginal densities.
For $x > 0$,

$$\begin{aligned} f_X(x) &= \int_0^\infty x^2 e^{-x(y+1)}\, dy \\ &= x^2 e^{-x} \int_0^\infty e^{-xy}\, dy \\ &= x^2 e^{-x} \left(\frac{1}{x}\right) \\ &= xe^{-x}. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} xe^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $y \geq 0$,

$$f_Y(y) = \int_0^\infty x^2 e^{-(y+1)x}\, dx.$$

Using

$$\int_0^\infty x^2 e^{-ax}\, dx = \frac{2}{a^3}, \qquad a > 0,$$

we obtain

$$f_Y(y) = \frac{2}{(y+1)^3}, \qquad y \geq 0.$$

Therefore,

$$f_X(x) f_Y(y) = \frac{2xe^{-x}}{(y+1)^3}.$$

But the joint density is

$$f(x,y) = x^2 e^{-x(y+1)}.$$

These are not equal in general. Hence,

$$f(x,y) \neq f_X(x) f_Y(y),$$

and therefore X and Y are not independent.

Problem 12

Suppose that X and Y are independent, identically distributed exponential random variables with mean $1/\lambda$. Prove that

$$\frac{X}{X+Y}$$

is uniformly distributed on $(0,1)$.

Solution

Since

$$X, Y \overset{\text{i.i.d.}}{\sim} \operatorname{Exp}(\lambda),$$

their joint density is

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}, \qquad x > 0,\ y > 0.$$

Define

$$U = \frac{X}{X+Y}, \qquad V = X+Y.$$

Then

$$X = UV$$

and

$$Y = (1-U)V.$$

Since $X > 0$ and $Y > 0$, the support of (U,V) is

$$0 < u < 1, \qquad v > 0.$$

The Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} v & u \\ -v & 1-u \end{pmatrix}.$$

Therefore,

$$\det \frac{\partial(x,y)}{\partial(u,v)} = v(1-u) + uv = v.$$

Hence,

$$\left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| = v.$$

The joint density of U and V is therefore

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, (1-u)v)v \\ &= \lambda^2 e^{-\lambda v} v, \end{aligned}$$

for

$$0 < u < 1, \qquad v > 0.$$

Now integrate out V:

$$\begin{aligned} f_U(u) &= \int_0^\infty \lambda^2 v e^{-\lambda v}\, dv \\ &= 1, \qquad 0 < u < 1. \end{aligned}$$

Thus,

$$f_U(u) = \begin{cases} 1, & 0 < u < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{X}{X+Y} \sim \text{Unif}(0,1).$$

Problem 13

Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{25}(x^2+y^2), & x = 1,2, \quad y = 0,1,2, \\ 0, & \text{otherwise.} \end{cases}$$

Find

(a) $p_{X|Y}(x \mid y)$;

(b) $P(X = 2 \mid Y = 1)$;

(c) $E(X \mid Y = 1)$.

Solution

First find the marginal PMF of Y. For $y = 0, 1, 2$,

$$\begin{aligned} p_Y(y) &= \sum_{x=1}^{2} p(x,y) \\ &= \frac{1}{25}\left[(1+y^2)+(4+y^2)\right] \\ &= \frac{5+2y^2}{25}. \end{aligned}$$

(a) By definition,

$$p_{X|Y}(x \mid y) = \frac{p(x,y)}{p_Y(y)}.$$

Therefore,

$$p_{X|Y}(x \mid y) = \frac{x^2+y^2}{5+2y^2}, \qquad x = 1,2, \quad y = 0,1,2.$$

(b) When $Y = 1$,

$$P(X = 2 \mid Y = 1) = \frac{2^2+1^2}{5+2(1)^2} = \frac{5}{7}.$$

(c) When $Y = 1$,

$$P(X = 1 \mid Y = 1) = \frac{1^2+1^2}{7} = \frac{2}{7},$$

and

$$P(X = 2 \mid Y = 1) = \frac{5}{7}.$$

Hence,

$$\begin{aligned} E(X \mid Y = 1) &= 1\left(\frac{2}{7}\right) + 2\left(\frac{5}{7}\right) \\ &= \frac{12}{7}. \end{aligned}$$

Problem 14

Let the conditional probability density function of X given $Y = y$ be

$$f_{X|Y}(x \mid y) = \frac{3(x^2 + y^2)}{3y^2 + 1}, \qquad 0 < x < 1, \quad 0 < y < 1.$$

Find

$$P\left(\frac{1}{4} < X < \frac{1}{2} \,\middle|\, Y = \frac{3}{4}\right).$$

Solution

Substituting

$$y = \frac{3}{4}$$

into the conditional density gives

$$f_{X|Y}\left(x \mid \frac{3}{4}\right) = \frac{3\left(x^2 + \frac{9}{16}\right)}{3\left(\frac{9}{16}\right) + 1}.$$

Since

$$3\left(\frac{9}{16}\right) + 1 = \frac{43}{16},$$

we obtain

$$f_{X|Y}\left(x \mid \frac{3}{4}\right) = \frac{48}{43}\left(x^2 + \frac{9}{16}\right).$$

Therefore,

$$\begin{aligned}
P\left(\frac{1}{4} < X < \frac{1}{2} \,\middle|\, Y = \frac{3}{4}\right) &= \int_{1/4}^{1/2} f_{X|Y}\left(x \mid \frac{3}{4}\right) dx \\
&= \frac{48}{43}\int_{1/4}^{1/2}\left(x^2 + \frac{9}{16}\right) dx \\
&= \frac{48}{43}\left[\frac{x^3}{3} + \frac{9}{16}x\right]_{1/4}^{1/2} \\
&= \frac{17}{86}.
\end{aligned}$$

Thus,

$$P\left(\frac{1}{4} < X < \frac{1}{2} \,\middle|\, Y = \frac{3}{4}\right) = \frac{17}{86}.$$

Problem 15

Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate $f_{X|Y}(x \mid y)$.

Solution

First calculate the marginal density of Y:

$$\begin{aligned} f_Y(y) &= \int_0^1 (x+y)\,dx \\ &= \left[\frac{x^2}{2} + xy\right]_0^1 \\ &= \frac{1}{2} + y, \end{aligned}$$

for

$$0 \le y \le 1.$$

Therefore,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{x+y}{y+\frac{1}{2}}, \end{aligned}$$

for

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

Thus,

$$f_{X|Y}(x \mid y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}}, & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 16

The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} ce^{-x}, & x \geq 0, \quad |y| < x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Find $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$.

(c) Calculate $E(Y \mid X = x)$ and $\mathrm{Var}(Y \mid X = x)$.

Solution

The support is

$$x \geq 0, \qquad -x < y < x.$$

(a) The density must integrate to 1:

$$\int_0^\infty \int_{-x}^{x} ce^{-x}\, dy\, dx = 1.$$

Therefore,

$$\begin{aligned} 1 &= c\int_0^\infty 2xe^{-x}\, dx \\ &= 2c. \end{aligned}$$

Hence,

$$c = \frac{1}{2}.$$

Thus,

$$f(x,y) = \frac{1}{2}e^{-x}$$

on the support.

(b) First,

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} \frac{1}{2}e^{-x}\, dy \\ &= xe^{-x}, \qquad x > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{Y|X}(y \mid x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{2}e^{-x}}{xe^{-x}} \\ &= \frac{1}{2x}, \end{aligned}$$

for

$$-x < y < x.$$

Thus,

$$f_{Y|X}(y \mid x) = \begin{cases} \dfrac{1}{2x}, & -x < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, conditional on $X = x$,

$$Y \mid X = x \sim \text{Unif}(-x, x).$$

Next, for a fixed y, the condition

$$|y| < x$$

implies

$$x > |y|.$$

Therefore,

$$\begin{aligned} f_Y(y) &= \int_{|y|}^{\infty} \frac{1}{2} e^{-x}\, dx \\ &= \frac{1}{2} e^{-|y|}. \end{aligned}$$

Hence,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{1}{2}e^{-x}}{\frac{1}{2}e^{-|y|}} \\ &= e^{-(x-|y|)}, \end{aligned}$$

for

$$x > |y|.$$

Thus,

$$f_{X|Y}(x \mid y) = \begin{cases} e^{-(x-|y|)}, & x > |y|, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Since

$$Y \mid X = x \sim \text{Unif}(-x, x),$$

symmetry gives

$$E(Y \mid X = x) = 0.$$

The variance of a uniform random variable on (a, b) is

$$\frac{(b-a)^2}{12}.$$

Therefore,

$$\begin{aligned} \text{Var}(Y \mid X = x) &= \frac{(x-(-x))^2}{12} \\ &= \frac{4x^2}{12} \\ &= \frac{x^2}{3}. \end{aligned}$$

Problem 17

Let X and Y be independent random variables, each uniformly distributed on $(0,1)$. Find the joint probability density function of

$$U = -2\ln X \qquad \text{and} \qquad V = -2\ln Y.$$

Solution

Since

$$X, Y \overset{\text{i.i.d.}}{\sim} \text{Unif}(0,1),$$

their joint density is

$$f_{X,Y}(x,y) = 1, \qquad 0 < x < 1, \quad 0 < y < 1.$$

The transformation is

$$u = -2\ln x, \qquad v = -2\ln y.$$

Solving for x and y,

$$x = e^{-u/2}, \qquad y = e^{-v/2}.$$

Since $0 < x, y < 1$, we have

$$u > 0, \qquad v > 0.$$

The Jacobian matrix of the inverse transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} -\frac{1}{2}e^{-u/2} & 0 \\ 0 & -\frac{1}{2}e^{-v/2} \end{pmatrix}.$$

Therefore,

$$\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{4}e^{-(u+v)/2}.$$

Hence,

$$f_{U,V}(u,v) = f_{X,Y}\left(e^{-u/2}, e^{-v/2}\right)\frac{1}{4}e^{-(u+v)/2}.$$

Since the original joint density equals 1 on its support,

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{4}e^{-(u+v)/2}, & u > 0, \quad v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 18

Let X and Y be two positive independent continuous random variables with probability density functions $f_1(x)$ and $f_2(y)$, respectively. Find the probability density function of

$$U = \frac{X}{Y}.$$

Hint: Let $V = X$; find the joint probability density function of U and V, and then calculate the marginal probability density function of U.

Solution

Define

$$U = \frac{X}{Y}, \qquad V = X.$$

Then

$$u = \frac{x}{y}, \qquad v = x.$$

Solving for x and y,

$$x = v, \qquad y = \frac{v}{u}.$$

Since X and Y are positive,

$$u > 0, \qquad v > 0.$$

The Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 0 & 1 \\ -\dfrac{v}{u^2} & \dfrac{1}{u} \end{pmatrix}.$$

Therefore,

$$\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{v}{u^2}.$$

Since X and Y are independent,

$$f_{X,Y}(x,y) = f_1(x) f_2(y).$$

Thus,

$$f_{U,V}(u,v) = f_1(v) f_2\left(\frac{v}{u}\right)\frac{v}{u^2}, \qquad u > 0, \quad v > 0.$$

Integrating out V,

$$f_U(u) = \int_0^\infty f_1(v) f_2\left(\frac{v}{u}\right)\frac{v}{u^2}\, dv, \qquad u > 0.$$

Therefore,

$$f_U(u) = \begin{cases} \displaystyle\int_0^\infty \frac{v}{u^2} f_1(v) f_2\left(\frac{v}{u}\right) dv, & u > 0, \\ 0, & u \le 0. \end{cases}$$

Problem 19

Let X and Y be independent random variables with common probability density function

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & x \geq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate the joint probability density function of

$$U = \frac{X}{Y} \qquad \text{and} \qquad V = XY.$$

Solution

The transformation is

$$u = \frac{x}{y}, \qquad v = xy.$$

Since $x > 0$ and $y > 0$,

$$uv = x^2,$$

and

$$\frac{v}{u} = y^2.$$

Therefore,

$$x = \sqrt{uv}, \qquad y = \sqrt{\frac{v}{u}}.$$

The Jacobian matrix of the inverse transformation is

$$\frac{\partial(x, y)}{\partial(u, v)}.$$

Its absolute determinant is

$$\left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2u}.$$

Since X and Y are independent,

$$f_{X,Y}(x, y) = \frac{1}{x^2 y^2}, \qquad x \geq 1, \quad y \geq 1.$$

But

$$x^2 y^2 = (xy)^2 = v^2.$$

Therefore,

$$f_{X,Y}(x, y) = \frac{1}{v^2}.$$

It remains to determine the support.
The condition $x \geq 1$ gives

$$\sqrt{uv} \geq 1,$$

so

$$uv \geq 1$$

and therefore

$$v \geq \frac{1}{u}.$$

The condition $y \geq 1$ gives

$$\sqrt{\frac{v}{u}} \geq 1,$$

so

$$v \geq u.$$

Thus,

$$u > 0, \qquad v \geq \max\left\{u, \frac{1}{u}\right\}.$$

Hence,

$$f_{U,V}(u,v) = \begin{cases} \dfrac{1}{2uv^2}, & u > 0, \quad v \geq \max\left\{u, \dfrac{1}{u}\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 20

Let X and Y be independent random variables with common probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the joint probability density function of

$$U = X + Y \qquad \text{and} \qquad V = e^X.$$

Solution

The transformation is

$$u = x + y, \qquad v = e^x.$$

From $v = e^x$, we obtain $x = \ln v$. Therefore, $y = u - \ln v$. Since $x > 0$, $\ln v > 0$, so $v > 1$. Since $y > 0$, $u - \ln v > 0$, so $u > \ln v$.

The Jacobian matrix of the inverse transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{1}{v} \end{pmatrix}.$$

Therefore,

$$\left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{v}.$$

Because X and Y are independent,

$$f_{X,Y}(x, y) = e^{-x}e^{-y} = e^{-(x+y)}.$$

Since

$$x + y = u,$$

we have

$$f_{X,Y}(x, y) = e^{-u}.$$

Hence,

$$f_{U,V}(u, v) = \begin{cases} \dfrac{e^{-u}}{v}, & v > 1, \quad u > \ln v, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 21

Suppose that X and Y are independent standard Normal random variables. Prove that

$$X + Y \qquad \text{and} \qquad X - Y$$

are independent random variables.

Solution

Define

$$U = X + Y, \qquad V = X - Y.$$

Solving for X and Y,

$$X = \frac{U+V}{2}, \qquad Y = \frac{U-V}{2}.$$

The Jacobian matrix of the inverse transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Therefore,

$$\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

Since X and Y are independent standard Normal random variables,

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right).$$

Now

$$\begin{aligned} x^2 + y^2 &= \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 \\ &= \frac{u^2+v^2}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi} \exp\left(-\frac{u^2+v^2}{4}\right) \frac{1}{2} \\ &= \frac{1}{4\pi} e^{-u^2/4} e^{-v^2/4}. \end{aligned}$$

But

$$\frac{1}{\sqrt{4\pi}} e^{-u^2/4}$$

is the density of a $N(0,2)$ random variable. Hence,

$$f_U(u) = \frac{1}{\sqrt{4\pi}} e^{-u^2/4}$$

and

$$f_V(v) = \frac{1}{\sqrt{4\pi}} e^{-v^2/4}.$$

Consequently, $f_{U,V}(u,v) = f_U(u) f_V(v)$ and thus the independence.

Problem 22

Let X and Y be independent strictly positive exponential random variables, each with parameter λ. Are the random variables

$$X + Y \qquad \text{and} \qquad \frac{X}{Y}$$

independent?

Solution

Define

$$U = X + Y, \qquad V = \frac{X}{Y}.$$

Since

$$v = \frac{x}{y},$$

we have

$$x = vy.$$

Substituting into

$$u = x + y$$

gives

$$u = (v+1)y.$$

Therefore,

$$y = \frac{u}{1+v}$$

and

$$x = \frac{uv}{1+v}.$$

Since $X, Y > 0$,

$$u > 0, \qquad v > 0.$$

The Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \dfrac{v}{1+v} & \dfrac{u}{(1+v)^2} \\ \dfrac{1}{1+v} & -\dfrac{u}{(1+v)^2} \end{pmatrix}.$$

Therefore,

$$\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{u}{(1+v)^2}.$$

Because X and Y are independent exponential random variables,

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}, \qquad x, y > 0.$$

Since

$$x + y = u,$$

the transformed joint density is

$$\begin{aligned} f_{U,V}(u,v) &= \lambda^2 e^{-\lambda u} \frac{u}{(1+v)^2} \\ &= \left(\lambda^2 u e^{-\lambda u}\right)\left(\frac{1}{(1+v)^2}\right), \end{aligned}$$

for

$$u > 0, \qquad v > 0.$$

The first factor is a density on $u > 0$, since

$$\int_0^\infty \lambda^2 u e^{-\lambda u}\, du = 1.$$

The second factor is also a density on $v > 0$, since

$$\int_0^\infty \frac{1}{(1+v)^2}\, dv = 1.$$

Thus,

$$f_U(u) = \lambda^2 u e^{-\lambda u}, \qquad u > 0,$$

and

$$f_V(v) = \frac{1}{(1+v)^2}, \qquad v > 0.$$

Therefore,

$$f_{U,V}(u,v) = f_U(u) f_V(v).$$

Hence,

$$X + Y \qquad \text{and} \qquad \frac{X}{Y}$$

are independent.

Problem 23

Let

$$p(x,y,z) = \frac{xyz}{162}, \qquad x = 4,5, \quad y = 1,2,3, \quad z = 1,2,$$

be the joint probability mass function of random variables X, Y, Z.

(a) Calculate the joint marginal probability mass functions of (X,Y), (Y,Z), and (X,Z).

(b) Find $E(YZ)$.

Solution

(a) To obtain the joint marginal PMF of X and Y, sum over all possible values of Z:

$$\begin{aligned} p_{X,Y}(x,y) &= \sum_{z=1}^{2} p(x,y,z) \\ &= \sum_{z=1}^{2} \frac{xyz}{162} \\ &= \frac{xy}{162}(1+2) \\ &= \frac{xy}{54}. \end{aligned}$$

Thus,

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{xy}{54}, & x = 4,5, \quad y = 1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$\begin{aligned} p_{Y,Z}(y,z) &= \sum_{x=4}^{5} p(x,y,z) \\ &= \frac{yz}{162}(4+5) \\ &= \frac{yz}{18}. \end{aligned}$$

Therefore,

$$p_{Y,Z}(y,z) = \begin{cases} \dfrac{yz}{18}, & y = 1,2,3, \quad z = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned} p_{X,Z}(x,z) &= \sum_{y=1}^{3} p(x,y,z) \\ &= \frac{xz}{162}(1+2+3) \\ &= \frac{xz}{27}. \end{aligned}$$

Hence,

$$p_{X,Z}(x,z) = \begin{cases} \dfrac{xz}{27}, & x = 4,5, \quad z = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Using the joint marginal PMF of Y and Z,

$$E(YZ) = \sum_{y=1}^{3}\sum_{z=1}^{2} yz\, p_{Y,Z}(y,z).$$

Thus,

$$\begin{aligned} E(YZ) &= \sum_{y=1}^{3}\sum_{z=1}^{2} yz\frac{yz}{18} \\ &= \frac{1}{18}\left(\sum_{y=1}^{3} y^2\right)\left(\sum_{z=1}^{2} z^2\right) \\ &= \frac{1}{18}(1+4+9)(1+4) \\ &= \frac{70}{18} \\ &= \frac{35}{9}. \end{aligned}$$

Therefore,

$$E(YZ) = \frac{35}{9}.$$

Problem 24

Let the joint probability density function of X, Y, Z be

$$f(x,y,z) = \begin{cases} 6e^{-x-y-z}, & 0 < x < y < z < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the joint marginal probability density functions of (X, Y), (X, Z), and (Y, Z).

(b) Find $E(X)$.

Solution

The support is

$$0 < x < y < z < \infty.$$

(a) For fixed x and y satisfying $0 < x < y$,

$$z > y.$$

Therefore,

$$\begin{aligned} f_{X,Y}(x,y) &= \int_y^\infty 6e^{-x-y-z}\,dz \\ &= 6e^{-x-y}\int_y^\infty e^{-z}\,dz \\ &= 6e^{-x-2y}. \end{aligned}$$

Hence,

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-x-2y}, & 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

For fixed x and z satisfying $0 < x < z$,

$$x < y < z.$$

Thus,

$$\begin{aligned} f_{X,Z}(x,z) &= \int_x^z 6e^{-x-y-z}\,dy \\ &= 6e^{-x-z}\left(e^{-x} - e^{-z}\right) \\ &= 6\left(e^{-2x-z} - e^{-x-2z}\right). \end{aligned}$$

Therefore,

$$f_{X,Z}(x,z) = \begin{cases} 6\left(e^{-2x-z} - e^{-x-2z}\right), & 0 < x < z, \\ 0, & \text{otherwise.} \end{cases}$$

For fixed y and z satisfying $0 < y < z$,

$$0 < x < y.$$

Hence,

$$\begin{aligned} f_{Y,Z}(y,z) &= \int_0^y 6e^{-x-y-z}\,dx \\ &= 6e^{-y-z}(1 - e^{-y}). \end{aligned}$$

Thus,

$$f_{Y,Z}(y,z) = \begin{cases} 6e^{-y-z}(1-e^{-y}), & 0 < y < z, \\ 0, & \text{otherwise.} \end{cases}$$

(b) We first find the marginal density of X:

$$\begin{aligned} f_X(x) &= \int_x^\infty \int_y^\infty 6e^{-x-y-z}\,dz\,dy \\ &= 6e^{-x}\int_x^\infty e^{-y}\left(\int_y^\infty e^{-z}\,dz\right)dy \\ &= 6e^{-x}\int_x^\infty e^{-2y}\,dy \\ &= 6e^{-x}\left(\frac{1}{2}e^{-2x}\right) \\ &= 3e^{-3x}, \qquad x > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} E(X) &= \int_0^\infty x\,3e^{-3x}\,dx \\ &= \frac{1}{3}. \end{aligned}$$

Thus,

$$E(X) = \frac{1}{3}.$$

Problem 25

Let X, Y, Z be jointly continuous with joint probability density function

$$f(x,y,z) = \begin{cases} x^2 e^{-x(1+y+z)}, & x,y,z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Are X, Y, Z mutually independent? Are they pairwise independent?

Solution

We first calculate the marginal densities.
For $x > 0$,

$$\begin{aligned} f_X(x) &= \int_0^\infty \int_0^\infty x^2 e^{-x(1+y+z)}\, dy\, dz \\ &= x^2 e^{-x} \left(\int_0^\infty e^{-xy}\, dy \right) \left(\int_0^\infty e^{-xz}\, dz \right) \\ &= x^2 e^{-x} \left(\frac{1}{x} \right) \left(\frac{1}{x} \right) \\ &= e^{-x}. \end{aligned}$$

For $y > 0$,

$$\begin{aligned} f_Y(y) &= \int_0^\infty \int_0^\infty x^2 e^{-x(1+y+z)}\, dz\, dx \\ &= \int_0^\infty x e^{-x(1+y)}\, dx \\ &= \frac{1}{(1+y)^2}. \end{aligned}$$

By symmetry,

$$f_Z(z) = \frac{1}{(1+z)^2}, \qquad z > 0.$$

If X, Y, Z were mutually independent, then

$$f(x,y,z) = f_X(x) f_Y(y) f_Z(z).$$

However,

$$f_X(x) f_Y(y) f_Z(z) = \frac{e^{-x}}{(1+y)^2 (1+z)^2},$$

which is not equal to

$$x^2 e^{-x(1+y+z)}.$$

Therefore, the three random variables are not mutually independent.
We now examine pairwise independence.
For $x, y > 0$,

$$\begin{aligned} f_{X,Y}(x,y) &= \int_0^\infty x^2 e^{-x(1+y+z)}\, dz \\ &= x e^{-x(1+y)}. \end{aligned}$$

But

$$f_X(x) f_Y(y) = \frac{e^{-x}}{(1+y)^2},$$

which is not equal to $f_{X,Y}(x,y)$. Hence X and Y are not independent.

Similarly, X and Z are not independent.
Finally,

$$\begin{aligned} f_{Y,Z}(y,z) &= \int_0^\infty x^2 e^{-x(1+y+z)}\,dx \\ &= \frac{2}{(1+y+z)^3}, \end{aligned}$$

whereas

$$f_Y(y)f_Z(z) = \frac{1}{(1+y)^2(1+z)^2}.$$

Thus Y and Z are not independent either.
Therefore, X, Y, Z are neither mutually independent nor pairwise independent.

Problem 26

Let the joint cumulative distribution function of X, Y, Z be

$$F(x,y,z) = \left(1 - e^{-\lambda_1 x}\right)\left(1 - e^{-\lambda_2 y}\right)\left(1 - e^{-\lambda_3 z}\right), \qquad x, y, z > 0,$$

where

$$\lambda_1, \lambda_2, \lambda_3 > 0.$$

(a) Are X, Y, Z independent?

(b) Find the joint probability density function of X, Y, Z.

(c) Find $P(X < Y < Z)$.

Solution

(a) The joint CDF factors as

$$F(x,y,z) = F_X(x)F_Y(y)F_Z(z),$$

where

$$F_X(x) = 1 - e^{-\lambda_1 x},$$
$$F_Y(y) = 1 - e^{-\lambda_2 y},$$

and

$$F_Z(z) = 1 - e^{-\lambda_3 z},$$

for positive arguments.

Therefore,

$$X \sim \text{Exp}(\lambda_1), \qquad Y \sim \text{Exp}(\lambda_2), \qquad Z \sim \text{Exp}(\lambda_3),$$

and X, Y, Z are mutually independent.

(b) Differentiating the joint CDF,

$$f(x,y,z) = \frac{\partial^3}{\partial x\,\partial y\,\partial z} F(x,y,z).$$

Therefore,

$$f(x,y,z) = \lambda_1\lambda_2\lambda_3 e^{-\lambda_1 x - \lambda_2 y - \lambda_3 z}, \qquad x, y, z > 0.$$

Thus,

$$f(x,y,z) = \begin{cases} \lambda_1\lambda_2\lambda_3 e^{-\lambda_1 x - \lambda_2 y - \lambda_3 z}, & x, y, z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We calculate

$$P(X < Y < Z) = \int_0^\infty \int_x^\infty \int_y^\infty f(x,y,z)\,dz\,dy\,dx.$$

Hence,

$$\begin{aligned}
P(X<Y<Z) &= \int_0^\infty \int_x^\infty \int_y^\infty \lambda_1\lambda_2\lambda_3 e^{-\lambda_1 x-\lambda_2 y-\lambda_3 z}\,dz\,dy\,dx \\
&= \int_0^\infty \int_x^\infty \lambda_1\lambda_2 e^{-\lambda_1 x-(\lambda_2+\lambda_3)y}\,dy\,dx \\
&= \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_3}\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)x}\,dx \\
&= \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}.
\end{aligned}$$

Therefore,

$$P(X<Y<Z) = \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)}.$$

Problem 27

Suppose that h is the probability density function of a continuous random variable. Let the joint probability density function of X, Y, Z be

$$f(x, y, z) = h(x)h(y)h(z), \qquad x, y, z \in \mathbb{R}.$$

Prove that

$$P(X < Y < Z) = \frac{1}{6}.$$

Solution

Since

$$f(x, y, z) = h(x)h(y)h(z),$$

the random variables X, Y, Z are independent and identically distributed.
Because their common distribution is continuous, ties occur with probability zero:

$$P(X = Y) = P(X = Z) = P(Y = Z) = 0.$$

Therefore, with probability 1, the three values occur in exactly one of the six possible strict orderings:

$$X < Y < Z,$$

$$X < Z < Y,$$

$$Y < X < Z,$$

$$Y < Z < X,$$

$$Z < X < Y,$$

or

$$Z < Y < X.$$

Since X, Y, Z are identically distributed, symmetry implies that all six orderings have the same probability.
Their probabilities sum to 1, so each must have probability

$$\frac{1}{6}.$$

Hence,

$$P(X < Y < Z) = \frac{1}{6}.$$

Problem 28

Let X and Y be random variables with finite second moments, and let a, b be constants. Prove that

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y).$$

Deduce that if X and Y are independent, then

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y).$$

Solution

By definition,

$$\operatorname{Var}(aX + bY) = E\left[(aX + bY) - E(aX + bY)\right]^2.$$

By linearity of expectation,

$$E(aX + bY) = aE(X) + bE(Y).$$

Therefore,

$$\begin{aligned}\operatorname{Var}(aX + bY) &= E\left[a(X - E(X)) + b(Y - E(Y))\right]^2 \\ &= E\Big[a^2(X - E(X))^2 + b^2(Y - E(Y))^2 \\ &\qquad + 2ab(X - E(X))(Y - E(Y))\Big].\end{aligned}$$

Using linearity of expectation,

$$\begin{aligned}\operatorname{Var}(aX + bY) = a^2 E[(X - E(X))^2] + b^2 E[(Y - E(Y))^2] \\ + 2abE[(X - E(X))(Y - E(Y))].\end{aligned}$$

Thus,

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y).$$

If X and Y are independent, then

$$\operatorname{Cov}(X, Y) = 0.$$

Hence,

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y).$$

Problem 29

Let $X_1, \ldots, X_n$ be random variables with finite second moments, and let $a_1, \ldots, a_n$ be constants. Prove that

$$\operatorname{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + 2\sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j).$$

In particular, show that

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{i<j} \operatorname{Cov}(X_i, X_j).$$

Solution

Let

$$S = \sum_{i=1}^{n} a_i X_i.$$

Then

$$E(S) = \sum_{i=1}^{n} a_i E(X_i).$$

Therefore,

$$S - E(S) = \sum_{i=1}^{n} a_i (X_i - E(X_i)).$$

Hence,

$$\operatorname{Var}(S) = E\left[\left(\sum_{i=1}^{n} a_i (X_i - E(X_i))\right)^2\right].$$

Expanding the square,

$$\operatorname{Var}(S) = E\left[\sum_{i=1}^{n} a_i^2 (X_i - E(X_i))^2 + 2\sum_{i<j} a_i a_j (X_i - E(X_i))(X_j - E(X_j))\right].$$

By linearity of expectation,

$$\operatorname{Var}(S) = \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + 2\sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j).$$

Taking

$$a_i = 1, \qquad i = 1, \ldots, n,$$

gives

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{i<j} \operatorname{Cov}(X_i, X_j).$$

Problem 30

Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Prove that

$$E(\overline{X}) = \mu$$

and

$$\operatorname{Var}(\overline{X}) = \frac{\sigma^2}{n}.$$

Solution

Since $X_1, \ldots, X_n$ form a random sample,

$$E(X_i) = \mu$$

and

$$\operatorname{Var}(X_i) = \sigma^2$$

for every i, and the random variables are independent.
By linearity of expectation,

$$\begin{aligned} E(\overline{X}) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\ &= \frac{1}{n}(n\mu) \\ &= \mu. \end{aligned}$$

For the variance, independence implies

$$\operatorname{Cov}(X_i, X_j) = 0 \qquad (i \neq j).$$

Hence,

$$\begin{aligned} \operatorname{Var}(\overline{X}) &= \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}(X_i) \\ &= \frac{1}{n^2}(n\sigma^2) \\ &= \frac{\sigma^2}{n}. \end{aligned}$$

Thus,

$$E(\overline{X}) = \mu, \qquad \operatorname{Var}(\overline{X}) = \frac{\sigma^2}{n}.$$

Problem 31

Let X be uniformly distributed on $(-1, 1)$, and define

$$Y = X^2.$$

Show that X and Y are dependent but uncorrelated.

Solution

Since

$$Y = X^2,$$

the value of Y is completely determined by X. Therefore, X and Y are not independent.
We now calculate their covariance.
By symmetry of the uniform distribution on $(-1, 1)$,

$$E(X) = 0.$$

Also,

$$E(X^3) = 0,$$

because x^3 is an odd function and the distribution of X is symmetric about zero.
Since

$$Y = X^2,$$

we have

$$XY = X^3.$$

Thus,

$$\begin{aligned}
\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= E(X^3) - E(X)E(X^2) \\
&= 0 - 0 \cdot E(X^2) \\
&= 0.
\end{aligned}$$

Therefore, X and Y are dependent but uncorrelated.

Problem 32

Let the joint probability mass function of random variables X and Y be

$$p(x,y) = \begin{cases} \frac{1}{70}x(x+y), & x = 1,2,3, \quad y = 3,4, \\ 0, & \text{elsewhere.} \end{cases}$$

Find

$$\operatorname{Cov}(X,Y).$$

Solution

Recall that

$$\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y).$$

First calculate $E(X)$:

$$\begin{aligned} E(X) &= \sum_{x=1}^{3}\sum_{y=3}^{4} x\,p(x,y) \\ &= \frac{1}{70}\sum_{x=1}^{3}\sum_{y=3}^{4} x^2(x+y). \end{aligned}$$

Evaluating the six terms,

$$E(X) = \frac{156}{70} = \frac{78}{35}.$$

Next,

$$\begin{aligned} E(Y) &= \sum_{x=1}^{3}\sum_{y=3}^{4} y\,p(x,y) \\ &= \frac{1}{70}\sum_{x=1}^{3}\sum_{y=3}^{4} xy(x+y) \\ &= \frac{249}{70}. \end{aligned}$$

Finally,

$$\begin{aligned} E(XY) &= \sum_{x=1}^{3}\sum_{y=3}^{4} xy\,p(x,y) \\ &= \frac{1}{70}\sum_{x=1}^{3}\sum_{y=3}^{4} x^2y(x+y) \\ &= \frac{558}{70} \\ &= \frac{279}{35}. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{Cov}(X,Y) &= \frac{279}{35} - \left(\frac{78}{35}\right)\left(\frac{249}{70}\right) \\ &= \frac{279}{35} - \frac{9711}{1225} \\ &= \frac{54}{1225}. \end{aligned}$$

Problem 33

Let X, Y, Z be random variables with finite second moments.

(a) Prove that
$$\operatorname{Cov}(X+Y, Z) = \operatorname{Cov}(X, Z) + \operatorname{Cov}(Y, Z).$$

(b) Prove that
$$\operatorname{Cov}(X, Y+Z) = \operatorname{Cov}(X, Y) + \operatorname{Cov}(X, Z).$$

(c) Deduce that
$$\operatorname{Cov}(X+Y, X-Y) = \operatorname{Var}(X) - \operatorname{Var}(Y).$$

(d) Deduce that
$$\operatorname{Var}(X-Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X, Y).$$

Solution

Recall that
$$\operatorname{Cov}(A, B) = E(AB) - E(A)E(B).$$

(a)
$$\begin{aligned}
\operatorname{Cov}(X+Y, Z) &= E[(X+Y)Z] - E(X+Y)E(Z) \\
&= E(XZ) + E(YZ) - [E(X) + E(Y)]E(Z) \\
&= [E(XZ) - E(X)E(Z)] \\
&\quad + [E(YZ) - E(Y)E(Z)] \\
&= \operatorname{Cov}(X, Z) + \operatorname{Cov}(Y, Z).
\end{aligned}$$

(b) Similarly,
$$\begin{aligned}
\operatorname{Cov}(X, Y+Z) &= E[X(Y+Z)] - E(X)E(Y+Z) \\
&= \operatorname{Cov}(X, Y) + \operatorname{Cov}(X, Z).
\end{aligned}$$

(c) Using bilinearity,
$$\begin{aligned}
\operatorname{Cov}(X+Y, X-Y) &= \operatorname{Cov}(X, X) - \operatorname{Cov}(X, Y) \\
&\quad + \operatorname{Cov}(Y, X) - \operatorname{Cov}(Y, Y).
\end{aligned}$$

Since
$$\operatorname{Cov}(X, X) = \operatorname{Var}(X),$$
$$\operatorname{Cov}(Y, Y) = \operatorname{Var}(Y),$$
and
$$\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X),$$
the middle terms cancel. Therefore,
$$\operatorname{Cov}(X+Y, X-Y) = \operatorname{Var}(X) - \operatorname{Var}(Y).$$

(d) Since
$$\operatorname{Var}(X-Y) = \operatorname{Cov}(X-Y, X-Y),$$

bilinearity gives

$$\begin{aligned}\operatorname{Var}(X-Y) &= \operatorname{Cov}(X,X) - \operatorname{Cov}(X,Y) \\ &\quad - \operatorname{Cov}(Y,X) + \operatorname{Cov}(Y,Y) \\ &= \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X,Y).\end{aligned}$$

Problem 34

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be random variables with finite second moments, and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be constants.

Prove that

$$\operatorname{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \operatorname{Cov}(X_i, Y_j).$$

Deduce, as a special case, that for constants a, b, c, d,

$$\operatorname{Cov}(aX + b, cY + d) = ac \operatorname{Cov}(X, Y).$$

Solution

Using bilinearity of covariance repeatedly,

$$\begin{aligned}
\operatorname{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) &= \sum_{i=1}^{n} a_i \operatorname{Cov}\left(X_i, \sum_{j=1}^{m} b_j Y_j\right) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j \operatorname{Cov}(X_i, Y_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \operatorname{Cov}(X_i, Y_j).
\end{aligned}$$

For the special case,

$$\operatorname{Cov}(aX + b, cY + d),$$

constants have zero covariance with every random variable. Hence,

$$\operatorname{Cov}(aX + b, cY + d) = ac \operatorname{Cov}(X, Y).$$

Problem 35

Let A and B be events with

$$P(A) > 0, \qquad P(B) > 0.$$

Let I_A and I_B denote their indicator random variables:

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$I_B = \begin{cases} 1, & \text{if } B \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Show that I_A and I_B are positively correlated if and only if

$$P(A \mid B) > P(A),$$

and equivalently if and only if

$$P(B \mid A) > P(B).$$

Solution

We have

$$E(I_A) = P(A), \qquad E(I_B) = P(B).$$

Also,

$$I_A I_B = I_{A \cap B},$$

so

$$E(I_A I_B) = P(A \cap B).$$

Therefore,

$$\mathrm{Cov}(I_A, I_B) = P(A \cap B) - P(A)P(B).$$

The indicators are positively correlated if and only if

$$\mathrm{Cov}(I_A, I_B) > 0.$$

Thus,

$$P(A \cap B) > P(A)P(B).$$

Since $P(B) > 0$, dividing by $P(B)$ gives

$$\frac{P(A \cap B)}{P(B)} > P(A),$$

or

$$P(A \mid B) > P(A).$$

Similarly, since $P(A) > 0$,

$$P(A \cap B) > P(A)P(B)$$

is equivalent to

$$\frac{P(A \cap B)}{P(A)} > P(B),$$

which is

$$P(B \mid A) > P(B).$$

Hence,

$$\operatorname{Cov}(I_A, I_B) > 0$$

if and only if

$$P(A \mid B) > P(A),$$

and if and only if

$$P(B \mid A) > P(B).$$

Problem 36

Let X and Y be integrable random variables.

Prove the **Law of Total Expectation**, also called the **Tower Property**:

$$E[E(X \mid Y)] = E(X).$$

Give a proof for the jointly continuous case.

Solution

Suppose X and Y have joint density $f_{X,Y}(x, y)$.
By definition,

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx.$$

Therefore,

$$\begin{aligned} E[E(X \mid Y)] &= \int_{-\infty}^{\infty} E(X \mid Y = y) f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx\right] f_Y(y)\, dy. \end{aligned}$$

Interchanging the order of integration,

$$E[E(X \mid Y)] = \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y)\, dy\right] dx.$$

Since

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)},$$

we have

$$f_{X|Y}(x \mid y) f_Y(y) = f_{X,Y}(x, y).$$

Hence,

$$\begin{aligned} E[E(X \mid Y)] &= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy\right] dx \\ &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= E(X). \end{aligned}$$

Therefore,

$$E[E(X \mid Y)] = E(X).$$

Problem 37

Let X and Y be random variables with $E(X^2) < \infty$.

Prove the **Law of Total Variance**:

$$\text{Var}(X) = E[\text{Var}(X \mid Y)] + \text{Var}(E(X \mid Y)) \,.$$

Solution

Recall that

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

For each value of Y,

$$\text{Var}(X \mid Y) = E(X^2 \mid Y) - [E(X \mid Y)]^2.$$

Taking expectations,

$$E[\text{Var}(X \mid Y)] = E[E(X^2 \mid Y)] - E\left([E(X \mid Y)]^2\right).$$

By the Law of Total Expectation,

$$E[E(X^2 \mid Y)] = E(X^2).$$

Therefore,

$$E[\text{Var}(X \mid Y)] = E(X^2) - E\left([E(X \mid Y)]^2\right).$$

Also,

$$\text{Var}(E(X \mid Y)) = E\left([E(X \mid Y)]^2\right) - (E[E(X \mid Y)])^2.$$

Again using the Law of Total Expectation,

$$E[E(X \mid Y)] = E(X).$$

Thus,

$$\text{Var}(E(X \mid Y)) = E\left([E(X \mid Y)]^2\right) - [E(X)]^2.$$

Adding the two expressions,

$$\begin{aligned}
&E[\text{Var}(X \mid Y)] + \text{Var}(E(X \mid Y)) \\
&= E(X^2) - E\left([E(X \mid Y)]^2\right) + E\left([E(X \mid Y)]^2\right) - [E(X)]^2 \\
&= E(X^2) - [E(X)]^2 \\
&= \text{Var}(X).
\end{aligned}$$

Therefore,

$$\text{Var}(X) = E[\text{Var}(X \mid Y)] + \text{Var}(E(X \mid Y)).$$

Problem 38

Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} \dfrac{3}{2}(x^2+y^2), & 0<x<1, \quad 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the random variable

$$E(X \mid Y).$$

Solution

We first find the marginal density of Y.
For $0<y<1$,

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{3}{2}(x^2+y^2)\,dx \\ &= \frac{3}{2}\left(\frac{1}{3}+y^2\right) \\ &= \frac{1}{2}+\frac{3}{2}y^2 \\ &= \frac{3y^2+1}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{3}{2}(x^2+y^2)}{\frac{1}{2}(3y^2+1)} \\ &= \frac{3(x^2+y^2)}{3y^2+1}, \end{aligned}$$

for $0<x<1$.
Therefore,

$$\begin{aligned} E(X \mid Y=y) &= \int_0^1 x f_{X|Y}(x \mid y)\,dx \\ &= \frac{3}{3y^2+1}\int_0^1 (x^3+xy^2)\,dx \\ &= \frac{3}{3y^2+1}\left(\frac{1}{4}+\frac{y^2}{2}\right) \\ &= \frac{3(2y^2+1)}{4(3y^2+1)}. \end{aligned}$$

Hence the conditional expectation as a random variable is

$$E(X \mid Y) = \frac{3(2Y^2+1)}{4(3Y^2+1)}.$$

Problem 39

For given random variables Y and Z, suppose that X is selected according to

$$X = \begin{cases} Y, & \text{with probability } p, \\ Z, & \text{with probability } 1-p, \end{cases}$$

where the random choice determining whether Y or Z is selected is independent of Y and Z.

Find $E(X)$ in terms of $E(Y)$ and $E(Z)$.

Solution

Let I be the indicator of the event that Y is selected. Then

$$P(I=1) = p, \qquad P(I=0) = 1-p,$$

and

$$X = IY + (1-I)Z.$$

Since I is independent of Y and Z,

$$E(IY) = E(I)E(Y) = pE(Y),$$

and

$$E[(1-I)Z] = E(1-I)E(Z) = (1-p)E(Z).$$

Therefore,

$$\begin{aligned} E(X) &= E[IY + (1-I)Z] \\ &= pE(Y) + (1-p)E(Z). \end{aligned}$$

Hence,

$$E(X) = pE(Y) + (1-p)E(Z).$$

Problem 40

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the moment-generating function $M_X(t)$.

(b) Using $M_X(t)$, find $E(X)$.

Solution

(a) By definition,

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

Therefore,

$$M_X(t) = 6 \int_0^1 x(1-x)e^{tx}\, dx.$$

Writing

$$x(1-x) = x - x^2,$$

we have

$$M_X(t) = 6 \left[\int_0^1 x e^{tx}\, dx - \int_0^1 x^2 e^{tx}\, dx \right].$$

After integration by parts,

$$M_X(t) = \frac{6\left[t + (t-2)e^t + 2\right]}{t^3}, \qquad t \neq 0.$$

At $t = 0$,

$$M_X(0) = 1.$$

Thus,

$$M_X(t) = \begin{cases} \dfrac{6\left[t + (t-2)e^t + 2\right]}{t^3}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

(b) Recall that

$$E(X) = M_X'(0).$$

To evaluate this derivative conveniently, expand $M_X(t)$ around $t = 0$. Using

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots,$$

we obtain

$$M_X(t) = 1 + \frac{1}{2}t + \frac{3}{20}t^2 + \cdots.$$

Therefore,

$$M_X'(0) = \frac{1}{2}.$$

Hence,

$$E(X) = \frac{1}{2}.$$

Problem 41

Let X be uniformly distributed on the interval (a, b), where $a < b$. Find the moment-generating function of X.

Solution

Since

$$X \sim \text{Unif}(a, b),$$

its density is

$$f_X(x) = \frac{1}{b-a}, \qquad a < x < b.$$

Therefore,

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \frac{1}{b-a} \int_a^b e^{tx}\, dx. \end{aligned}$$

For $t \neq 0$,

$$\begin{aligned} M_X(t) &= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b \\ &= \frac{e^{bt} - e^{at}}{(b-a)t}. \end{aligned}$$

Also,

$$M_X(0) = 1.$$

Thus,

$$M_X(t) = \begin{cases} \dfrac{e^{bt} - e^{at}}{(b-a)t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Problem 42

Let X be a geometric random variable with parameter p, using the convention

$$P(X = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots.$$

Let

$$q = 1 - p.$$

(a) Show that

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \qquad t < -\ln q.$$

(b) Use $M_X(t)$ to find $E(X)$ and $\mathrm{Var}(X)$.

Solution

(a) By definition,

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{k=1}^{\infty} e^{tk} pq^{k-1}. \end{aligned}$$

Factor out pe^t:

$$M_X(t) = pe^t \sum_{k=1}^{\infty} (qe^t)^{k-1}.$$

The geometric series converges when

$$|qe^t| < 1.$$

Since $q > 0$, this is equivalent to

$$t < -\ln q.$$

Therefore,

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \qquad t < -\ln q.$$

(b) Differentiate:

$$M_X'(t) = \frac{pe^t}{(1 - qe^t)^2}.$$

Hence,

$$E(X) = M_X'(0) = \frac{p}{(1-q)^2}.$$

Since

$$1 - q = p,$$

we obtain

$$E(X) = \frac{1}{p}.$$

Differentiate again:

$$M_X''(t) = \frac{pe^t(1 + qe^t)}{(1 - qe^t)^3}.$$

Therefore,

$$E(X^2) = M_X''(0) = \frac{1+q}{p^2}.$$

Thus,

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{1+q}{p^2} - \frac{1}{p^2} \\ &= \frac{q}{p^2}.\end{aligned}$$

Since $q = 1 - p$,

$$\mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Problem 43

Suppose that

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1,$$

is the moment-generating function of a random variable X.

Find the moment-generating function of

$$Y = 2X + 1.$$

Solution

By definition,

$$M_Y(t) = E(e^{tY}).$$

Since

$$Y = 2X + 1,$$

we have

$$\begin{aligned} M_Y(t) &= E\left(e^{t(2X+1)}\right) \\ &= e^t E(e^{2tX}) \\ &= e^t M_X(2t). \end{aligned}$$

Therefore,

$$M_Y(t) = e^t \frac{1}{1-2t}.$$

The MGF $M_X(2t)$ is finite provided

$$2t < 1,$$

that is,

$$t < \frac{1}{2}.$$

Hence,

$$M_Y(t) = \frac{e^t}{1-2t}, \qquad t < \frac{1}{2}.$$

Problem 44

For a random variable X,

$$M_X(t) = \left(\frac{2}{2-t}\right)^3.$$

Find $E(X)$ and $\mathrm{Var}(X)$.

Solution

Write

$$M_X(t) = 8(2-t)^{-3}.$$

Differentiating,

$$M_X'(t) = 24(2-t)^{-4}.$$

Therefore,

$$E(X) = M_X'(0) = \frac{24}{16} = \frac{3}{2}.$$

Differentiating again,

$$M_X''(t) = 96(2-t)^{-5}.$$

Thus,

$$E(X^2) = M_X''(0) = \frac{96}{32} = 3.$$

Hence,

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 3 - \left(\frac{3}{2}\right)^2 \\ &= 3 - \frac{9}{4} \\ &= \frac{3}{4}.\end{aligned}$$

Therefore,

$$E(X) = \frac{3}{2}, \qquad \mathrm{Var}(X) = \frac{3}{4}.$$

Problem 45

Let

$$Z \sim N(0,1).$$

Using

$$M_Z(t) = e^{t^2/2},$$

calculate

$$E(Z^n),$$

where n is a positive integer.

Solution

The moment-generating function has the power-series representation

$$M_Z(t) = \sum_{n=0}^{\infty} \frac{E(Z^n)}{n!} t^n.$$

On the other hand,

$$M_Z(t) = e^{t^2/2}.$$

Expanding the exponential,

$$\begin{aligned} e^{t^2/2} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t^2}{2}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}. \end{aligned}$$

There are no odd powers of t. Therefore,

$$E(Z^n) = 0$$

whenever n is odd.
If

$$n = 2k,$$

then comparison of the coefficient of t^{2k} gives

$$\frac{E(Z^{2k})}{(2k)!} = \frac{1}{2^k k!}.$$

Thus,

$$E(Z^{2k}) = \frac{(2k)!}{2^k k!}.$$

Equivalently,

$$E(Z^{2k}) = (2k-1)!!.$$

Therefore,

$$E(Z^n) = \begin{cases} 0, & n \text{ odd}, \\ \dfrac{n!}{2^{n/2}(n/2)!}, & n \text{ even}. \end{cases}$$

Problem 46

Let

$$X_1, \dots, X_n$$

be independent random variables with

$$X_i \sim N(\mu_i, \sigma_i^2), \qquad i = 1, \dots, n.$$

Let

$$a_1, \dots, a_n$$

be constants.

Using moment-generating functions, prove that

$$\sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right).$$

Deduce that if

$$X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2),$$

then

$$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Solution

For

$$X_i \sim N(\mu_i, \sigma_i^2),$$

the MGF is

$$M_{X_i}(t) = \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right).$$

Let

$$S = \sum_{i=1}^n a_i X_i.$$

Then

$$M_S(t) = E\left[e^{t\sum_{i=1}^n a_i X_i}\right].$$

Since the X_i's are independent,

$$\begin{aligned} M_S(t) &= \prod_{i=1}^n E(e^{t a_i X_i}) \\ &= \prod_{i=1}^n M_{X_i}(a_i t). \end{aligned}$$

Therefore,

$$\begin{aligned} M_S(t) &= \prod_{i=1}^n \exp\left(a_i \mu_i t + \frac{1}{2} a_i^2 \sigma_i^2 t^2\right) \\ &= \exp\left[t\sum_{i=1}^n a_i \mu_i + \frac{t^2}{2}\sum_{i=1}^n a_i^2 \sigma_i^2\right]. \end{aligned}$$

This is the MGF of a Normal random variable with mean

$$\sum_{i=1}^{n} a_i \mu_i$$

and variance

$$\sum_{i=1}^{n} a_i^2 \sigma_i^2.$$

Hence,

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

For the sample mean, take

$$a_i = \frac{1}{n}, \qquad \mu_i = \mu, \qquad \sigma_i^2 = \sigma^2.$$

Then

$$E(\overline{X}) = \sum_{i=1}^{n} \frac{1}{n}\mu = \mu,$$

and

$$\mathrm{Var}(\overline{X}) = \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n}.$$

Therefore,

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Problem 47

Let

$$X_1, X_2, \ldots, X_n$$

be independent geometric random variables, each with parameter p, using the convention

$$P(X_i = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \ldots.$$

Using moment-generating functions, prove that

$$X_1 + \cdots + X_n$$

has a negative binomial distribution with parameters (n, p).

Solution

Let

$$q = 1 - p.$$

The MGF of a geometric random variable under this convention is

$$M_{X_i}(t) = \frac{pe^t}{1 - qe^t}, \qquad t < -\ln q.$$

Define

$$S_n = X_1 + \cdots + X_n.$$

Since $X_1, \ldots, X_n$ are independent,

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(\frac{pe^t}{1 - qe^t} \right)^n.$$

Now recall that if

$$W \sim \mathrm{NegBin}(n, p),$$

where W denotes the trial on which the n-th success occurs, then

$$P(W = k) = \binom{k-1}{n-1} p^n q^{k-n}, \qquad k = n, n+1, \ldots.$$

Its MGF is

$$M_W(t) = \left(\frac{pe^t}{1 - qe^t} \right)^n.$$

Thus,

$$M_{S_n}(t) = M_W(t)$$

in a neighborhood of 0.
By uniqueness of moment-generating functions,

$$S_n = X_1 + \cdots + X_n \sim \mathrm{NegBin}(n, p).$$

Problem 48

Let X and Y be independent binomial random variables with

$$X \sim \mathrm{Bin}(n,p), \qquad Y \sim \mathrm{Bin}(m,p).$$

Calculate

$$P(X = i \mid X + Y = j)$$

and interpret the result.

Solution

Since X and Y are independent,

$$X + Y \sim \mathrm{Bin}(n+m,p).$$

Therefore,

$$P(X = i \mid X + Y = j) = \frac{P(X = i, Y = j - i)}{P(X + Y = j)}.$$

The numerator is

$$\begin{aligned} P(X = i, Y = j - i) &= P(X = i)P(Y = j - i) \\ &= \binom{n}{i} p^i (1-p)^{n-i} \\ &\quad \times \binom{m}{j-i} p^{j-i} (1-p)^{m-j+i}. \end{aligned}$$

Thus,

$$P(X = i, Y = j - i) = \binom{n}{i}\binom{m}{j-i} p^j (1-p)^{n+m-j}.$$

Also,

$$P(X + Y = j) = \binom{n+m}{j} p^j (1-p)^{n+m-j}.$$

Hence,

$$P(X = i \mid X + Y = j) = \frac{\binom{n}{i}\binom{m}{j-i}}{\binom{n+m}{j}},$$

for

$$\max\{0, j-m\} \le i \le \min\{n, j\}.$$

This is the probability mass function of a hypergeometric random variable.
Conditional on there being exactly j total successes among the $n+m$ Bernoulli trials, all choices of the j successful trials are equally likely. The random variable X counts how many of those j successes occur among the first n trials.
Therefore,

$$X \mid (X + Y = j) \sim \mathrm{Hypergeom}(n+m, n, j).$$

Problem 49

Let X, Y, Z be independent Poisson random variables with parameters

$$\lambda_1, \lambda_2, \lambda_3,$$

respectively.

For

$$y = 0, 1, \ldots, t,$$

calculate

$$P(Y = y \mid X + Y + Z = t).$$

Solution

Let

$$T = X + Y + Z.$$

Since independent Poisson random variables add,

$$T \sim \text{Pois}(\lambda_1 + \lambda_2 + \lambda_3).$$

Also,

$$X + Z \sim \text{Pois}(\lambda_1 + \lambda_3),$$

and $X + Z$ is independent of Y.
Therefore,

$$\begin{aligned} P(Y = y \mid T = t) &= \frac{P(Y = y,\ X + Z = t - y)}{P(T = t)} \\ &= \frac{P(Y = y)P(X + Z = t - y)}{P(T = t)}. \end{aligned}$$

Substituting the Poisson probabilities,

$$P(Y = y \mid T = t) = \frac{e^{-\lambda_2} \dfrac{\lambda_2^y}{y!} \, e^{-(\lambda_1+\lambda_3)} \dfrac{(\lambda_1 + \lambda_3)^{t-y}}{(t-y)!}}{e^{-(\lambda_1+\lambda_2+\lambda_3)} \dfrac{(\lambda_1 + \lambda_2 + \lambda_3)^t}{t!}}.$$

The exponential terms cancel, giving

$$P(Y = y \mid T = t) = \binom{t}{y} \frac{\lambda_2^y (\lambda_1 + \lambda_3)^{t-y}}{(\lambda_1 + \lambda_2 + \lambda_3)^t}.$$

Equivalently,

$$P(Y = y \mid T = t) = \binom{t}{y} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \right)^y \left(\frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \right)^{t-y}.$$

Thus,

$$Y \mid (X + Y + Z = t) \sim \text{Bin}\left(t, \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \right).$$

Problem 50

Let X and Y be i.i.d. positive random variables, and let $c > 0$.

For each part below, fill in the appropriate equality or inequality symbol.

Write $=$ if the two sides are always equal, $\leq$ if the left-hand side is less than or equal to the right-hand side but not necessarily equal, and similarly for $\geq$. If no relation holds in general, write ?.

(a)
$$E(\log X) \_\_\_\_\_ \log(EX).$$

(b)
$$E(X) \_\_\_\_\_ \sqrt{E(X^2)}.$$

(c)
$$E(\sin^2 X) + E(\cos^2 X) \_\_\_\_\_ 1.$$

(d)
$$E(|X|) \_\_\_\_\_ \sqrt{E(X^2)}.$$

(e)
$$P(X > c) \_\_\_\_\_ \frac{E(X^3)}{c^3}.$$

(f)
$$P(X \leq Y) \_\_\_\_\_ P(X \geq Y).$$

(g)
$$E(XY) \_\_\_\_\_ \sqrt{E(X^2)E(Y^2)}.$$

(h)
$$P(X + Y > 10) \_\_\_\_\_ P(X > 5 \text{ or } Y > 5).$$

(i)
$$E(\min(X, Y)) \_\_\_\_\_ \min(EX, EY).$$

(j)
$$E\left(\frac{X}{Y}\right) \_\_\_\_\_ \frac{EX}{EY}.$$

(k)
$$E\left(X^2(X^2+1)\right) \_\_\_\_\_ E\left(X^2(Y^2+1)\right).$$

(l)
$$E\left(\frac{X^3}{X^3+Y^3}\right) \_\_\_\_\_ E\left(\frac{Y^3}{X^3+Y^3}\right).$$

Solution

(a)
$$E(\log X) \leq \log(EX).$$

Since $\log x$ is concave on $(0,\infty)$, Jensen's inequality gives

$$E(\log X) \le \log(EX).$$

(b)

$$E(X) \le \sqrt{E(X^2)}.$$

By Cauchy–Schwarz,

$$[E(X)]^2 \le E(X^2).$$

Since $X > 0$,

$$E(X) \le \sqrt{E(X^2)}.$$

(c)

$$E(\sin^2 X) + E(\cos^2 X) = 1.$$

Indeed,

$$\sin^2 X + \cos^2 X = 1$$

for every value of X. Therefore,

$$\begin{aligned} E(\sin^2 X) + E(\cos^2 X) &= E(\sin^2 X + \cos^2 X) \\ &= E(1) \\ &= 1. \end{aligned}$$

(d)

$$E(|X|) \le \sqrt{E(X^2)}.$$

By Cauchy–Schwarz,

$$[E(|X|)]^2 \le E(|X|^2) = E(X^2).$$

Hence,

$$E(|X|) \le \sqrt{E(X^2)}.$$

(e)

$$P(X > c) \le \frac{E(X^3)}{c^3}.$$

Since $X > 0$,

$$X > c \iff X^3 > c^3.$$

Applying Markov's inequality to the nonnegative random variable X^3,

$$P(X > c) = P(X^3 > c^3) \le \frac{E(X^3)}{c^3}.$$

(f)

$$P(X \le Y) = P(X \ge Y).$$

Since X and Y are identically distributed and independent, the joint distribution of (X, Y) is unchanged when X and Y are exchanged.

Therefore,

$$P(X \le Y) = P(Y \le X) = P(X \ge Y).$$

(g)

$$E(XY) \le \sqrt{E(X^2)E(Y^2)}.$$

This follows directly from the Cauchy–Schwarz inequality:

$$|E(XY)| \le \sqrt{E(X^2)E(Y^2)}.$$

Since X and Y are positive,

$$E(XY) \ge 0,$$

and hence

$$E(XY) \le \sqrt{E(X^2)E(Y^2)}.$$

(h)

$$P(X + Y > 10) \le P(X > 5 \text{ or } Y > 5).$$

Indeed,

$$\{X + Y > 10\} \subseteq \{X > 5\} \cup \{Y > 5\}.$$

Therefore,

$$P(X + Y > 10) \le P(X > 5 \text{ or } Y > 5).$$

(i)

$$E(\min(X, Y)) \le \min(EX, EY).$$

Pointwise,

$$\min(X, Y) \le X$$

and

$$\min(X, Y) \le Y.$$

Taking expectations gives

$$E(\min(X, Y)) \le E(X)$$

and

$$E(\min(X, Y)) \le E(Y).$$

Therefore,

$$E(\min(X, Y)) \le \min(EX, EY).$$

(j)

$$E\left(\frac{X}{Y}\right) \ ? \ \frac{EX}{EY}.$$

No fixed relation holds in general.

By independence,

$$E\left(\frac{X}{Y}\right) = E(X)E\left(\frac{1}{Y}\right),$$

provided the expectations exist. This is generally not equal to

$$\frac{E(X)}{E(Y)}.$$

(k)
$$E\left(X^2(X^2+1)\right) \geq E\left(X^2(Y^2+1)\right).$$

For the left-hand side,
$$E\left(X^2(X^2+1)\right) = E(X^4) + E(X^2).$$

For the right-hand side, independence gives

$$\begin{aligned} E\left(X^2(Y^2+1)\right) &= E(X^2Y^2) + E(X^2) \\ &= E(X^2)E(Y^2) + E(X^2). \end{aligned}$$

Since X and Y are identically distributed,

$$E(Y^2) = E(X^2),$$

so
$$E\left(X^2(Y^2+1)\right) = [E(X^2)]^2 + E(X^2).$$

Finally,
$$E(X^4) \geq [E(X^2)]^2$$

because
$$\mathrm{Var}(X^2) \geq 0.$$

Therefore,
$$E\left(X^2(X^2+1)\right) \geq E\left(X^2(Y^2+1)\right).$$

(l)
$$E\left(\frac{X^3}{X^3+Y^3}\right) = E\left(\frac{Y^3}{X^3+Y^3}\right).$$

Since X and Y are i.i.d., exchanging X and Y does not change their joint distribution. Hence,

$$E\left(\frac{X^3}{X^3+Y^3}\right) = E\left(\frac{Y^3}{X^3+Y^3}\right).$$

In fact, since
$$\frac{X^3}{X^3+Y^3} + \frac{Y^3}{X^3+Y^3} = 1,$$

the two equal expectations must each be

$$\frac{1}{2}.$$

Problem 51

Let X be an exponential random variable with parameter 1, so that

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that

$$E(X) = 1, \qquad \operatorname{Var}(X) = 1,$$

and the moment generating function of X is

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1.$$

(a) Use Markov's inequality to find an upper bound for

$$P(X \geq 10).$$

(b) Use Chebyshev's inequality to find an upper bound for

$$P(X \geq 10).$$

(c) Recall that, for any $t > 0$,

$$P(X \geq a) \leq e^{-ta} M_X(t).$$

Use this Chernoff bound to show that

$$P(X \geq 10) \leq \frac{e^{-10t}}{1-t}, \qquad 0 < t < 1.$$

Find the value of t that gives the best bound, and calculate the resulting Chernoff bound.

(d) Compare the three bounds obtained above. Which inequality gives the sharpest upper bound for $P(X \geq 10)$?

Also calculate the exact value of $P(X \geq 10)$ and compare it with the three bounds.

(e) Compare

$$E\left(\frac{1}{1+X}\right) \qquad \text{and} \qquad \frac{1}{1+E(X)}$$

i.e., determine whether $=$, $\leq$, or $\geq$ always holds.

Hint: Consider the function

$$g(x) = \frac{1}{1+x}.$$

Check whether the conditions for Jensen's inequality are satisfied.

Solution

(a) Markov's inequality states that, for a nonnegative random variable,

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Since

$$E(X) = 1,$$

we obtain

$$P(X \geq 10) \leq \frac{1}{10}.$$

Thus, the Markov bound is

$$P(X \geq 10) \leq 0.1.$$

(b) To apply Chebyshev's inequality, observe that

$$X \geq 10$$

implies

$$X - E(X) \geq 9.$$

Therefore,

$$\{X \geq 10\} \subseteq \{|X - E(X)| \geq 9\}.$$

Hence,

$$\begin{aligned} P(X \geq 10) &\leq P(|X - E(X)| \geq 9) \\ &\leq \frac{\mathrm{Var}(X)}{9^2} \\ &= \frac{1}{81}. \end{aligned}$$

Thus, the Chebyshev bound is

$$P(X \geq 10) \leq \frac{1}{81} \approx 0.01235.$$

(c) For $t > 0$, the Chernoff bound gives

$$P(X \geq 10) \leq e^{-10t} M_X(t).$$

Since

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1,$$

we have

$$P(X \geq 10) \leq \frac{e^{-10t}}{1-t}, \qquad 0 < t < 1.$$

We now minimize

$$h(t) = \frac{e^{-10t}}{1-t}.$$

It is convenient to minimize its logarithm:

$$\log h(t) = -10t - \log(1-t).$$

Differentiating,

$$\frac{d}{dt} \log h(t) = -10 + \frac{1}{1-t}.$$

Setting this equal to zero,

$$-10 + \frac{1}{1-t} = 0.$$

Thus,

$$\frac{1}{1-t} = 10,$$

so

$$1 - t = \frac{1}{10},$$

and therefore

$$t = \frac{9}{10}.$$

Substituting this value into the Chernoff bound,

$$\begin{aligned} P(X \geq 10) &\leq \frac{e^{-10(9/10)}}{1 - 9/10} \\ &= \frac{e^{-9}}{1/10} \\ &= 10e^{-9}. \end{aligned}$$

Thus, the optimized Chernoff bound is

$$P(X \geq 10) \leq 10e^{-9} \approx 0.001234.$$

(d) The three bounds are

Method	Upper bound
Markov	$\frac{1}{10}$
Chebyshev	$\frac{1}{81}$
Chernoff	$10e^{-9}$

Numerically,

$$\frac{1}{10} = 0.1,$$

$$\frac{1}{81} \approx 0.01235,$$

and

$$10e^{-9} \approx 0.001234.$$

Therefore,

$$10e^{-9} < \frac{1}{81} < \frac{1}{10}.$$

Hence, among these three inequalities, the Chernoff bound gives the sharpest upper bound.

The exact probability can be calculated directly from the exponential distribution:

$$\begin{aligned} P(X \geq 10) &= \int_{10}^{\infty} e^{-x}\,dx \\ &= e^{-10}. \end{aligned}$$

Numerically,

$$e^{-10} \approx 0.0000454.$$

Thus,

$$\underbrace{e^{-10}}_{\text{exact}} < \underbrace{10e^{-9}}_{\text{Chernoff}} < \underbrace{\frac{1}{81}}_{\text{Chebyshev}} < \underbrace{\frac{1}{10}}_{\text{Markov}}.$$

(e) Define

$$g(x) = \frac{1}{1+x}, \qquad x > 0.$$

We have

$$g'(x) = -\frac{1}{(1+x)^2}$$

and

$$g''(x) = \frac{2}{(1+x)^3}.$$

Since

$$g''(x) > 0 \qquad \text{for } x > 0,$$

the function g is convex on the support of X.

Therefore, Jensen's inequality gives

$$g(E(X)) \le E(g(X)).$$

Hence,

$$\frac{1}{1+E(X)} \le E\left(\frac{1}{1+X}\right).$$

Equivalently,

$$E\left(\frac{1}{1+X}\right) \ge \frac{1}{1+E(X)}.$$

Since $E(X) = 1$,

$$E\left(\frac{1}{1+X}\right) \ge \frac{1}{2}.$$

Problem 52

Suppose that the number X of letters handled by a post office on a given day satisfies

$$E(X) = 10{,}000$$

and

$$\mathrm{Var}(X) = 2000.$$

Using Chebyshev's inequality, find a lower bound for

$$P(8000 < X < 12{,}000).$$

Solution

We can write

$$8000 < X < 12{,}000$$

as

$$-2000 < X - 10{,}000 < 2000.$$

Thus,

$$P(8000 < X < 12{,}000) = P(|X - 10{,}000| < 2000).$$

Using the complement,

$$P(|X - 10{,}000| < 2000) = 1 - P(|X - 10{,}000| \geq 2000).$$

By Chebyshev's inequality,

$$P(|X - 10{,}000| \geq 2000) \leq \frac{2000}{2000^2}.$$

Therefore,

$$P(|X - 10{,}000| \geq 2000) \leq 0.0005.$$

Hence,

$$P(8000 < X < 12{,}000) \geq 1 - 0.0005 = 0.9995.$$

Thus,

$$P(8000 < X < 12{,}000) \geq 0.9995.$$

Problem 53

Roll a fair six-sided die and let X denote the outcome.

(a) Calculate $E(X)$ and $\text{Var}(X)$.

(b) Use Markov's inequality to obtain an upper bound for

$$P(X \geq 6).$$

(c) Use Chebyshev's inequality to obtain an upper bound for

$$P\left(\left|X - \frac{21}{6}\right| \geq \frac{3}{2}\right).$$

(d) Calculate both probabilities exactly and compare the exact probabilities with the bounds.

Solution

(a) Since each outcome has probability $1/6$,

$$E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}.$$

Also,

$$E(X^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}.$$

Therefore,

$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{91}{6} - \frac{49}{4} \\ &= \frac{35}{12}.\end{aligned}$$

(b) By Markov's inequality,

$$P(X \geq 6) \leq \frac{E(X)}{6}.$$

Hence,

$$P(X \geq 6) \leq \frac{7/2}{6} = \frac{7}{12} \approx 0.5833.$$

(c) By Chebyshev's inequality,

$$P\left(\left|X - \frac{7}{2}\right| \geq \frac{3}{2}\right) \leq \frac{35/12}{(3/2)^2}.$$

Therefore,

$$P\left(\left|X - \frac{7}{2}\right| \geq \frac{3}{2}\right) \leq \frac{35}{27} \approx 1.296.$$

Since probabilities cannot exceed 1, this bound is trivial. The useful upper bound is simply

$$1.$$

(d) Exactly,

$$P(X \geq 6) = P(X = 6) = \frac{1}{6}.$$

Also,

$$\left| X - \frac{7}{2} \right| \geq \frac{3}{2}$$

occurs when

$$X \leq 2 \qquad \text{or} \qquad X \geq 5.$$

Thus,

$$P\left(\left| X - \frac{7}{2} \right| \geq \frac{3}{2} \right) = \frac{4}{6} = \frac{2}{3}.$$

Therefore, both inequalities give valid upper bounds, but the bounds may be far from the exact probabilities.

Problem 54

The scores on an achievement test have population mean

$$\mu = 500$$

and population standard deviation

$$\sigma = 100.$$

Let

$$\overline{X}$$

be the sample mean of a random sample of 10 students.

Without assuming that the population distribution is Normal, use Chebyshev's inequality to find a lower bound for

$$P(460 < \overline{X} < 540).$$

Solution

Since the sample consists of independent observations,

$$E(\overline{X}) = 500$$

and

$$\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n} = \frac{100^2}{10} = 1000.$$

Now,

$$\begin{aligned} P(460 < \overline{X} < 540) &= P(-40 < \overline{X} - 500 < 40) \\ &= P(|\overline{X} - 500| < 40). \end{aligned}$$

Using the complement,

$$P(|\overline{X} - 500| < 40) = 1 - P(|\overline{X} - 500| \geq 40).$$

By Chebyshev's inequality,

$$P(|\overline{X} - 500| \geq 40) \leq \frac{1000}{40^2}.$$

Thus,

$$P(|\overline{X} - 500| \geq 40) \leq \frac{1000}{1600} = \frac{5}{8}.$$

Therefore,

$$P(460 < \overline{X} < 540) \geq 1 - \frac{5}{8} = \frac{3}{8}.$$

Hence,

$$P(460 < \overline{X} < 540) \geq \frac{3}{8}.$$

Problem 55

Suppose

$$X_1, \dots, X_n$$

are independent Bernoulli random variables with unknown success probability p, and define

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Let

$$\varepsilon > 0 \qquad \text{and} \qquad 0 < \alpha < 1.$$

Using Chebyshev's inequality, derive a sample-size condition that guarantees

$$P(|\widehat{p} - p| < \varepsilon) \geq 1 - \alpha$$

for every

$$0 < p < 1.$$

Solution

Since

$$X_i \sim \text{Bernoulli}(p),$$

we have

$$E(X_i) = p$$

and

$$\text{Var}(X_i) = p(1-p).$$

Therefore,

$$E(\widehat{p}) = p,$$

and by independence,

$$\text{Var}(\widehat{p}) = \frac{p(1-p)}{n}.$$

By Chebyshev's inequality,

$$P(|\widehat{p} - p| \geq \varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2}.$$

Hence,

$$P(|\widehat{p} - p| < \varepsilon) \geq 1 - \frac{p(1-p)}{n\varepsilon^2}.$$

To guarantee

$$P(|\widehat{p} - p| < \varepsilon) \geq 1 - \alpha,$$

it is sufficient that

$$\frac{p(1-p)}{n\varepsilon^2} \leq \alpha.$$

Equivalently,

$$n \geq \frac{p(1-p)}{\varepsilon^2 \alpha}.$$

However, p is unknown. Since

$$p(1-p) \leq \frac{1}{4}$$

for every $0 < p < 1$, a sufficient condition that does not depend on p is

$$n \geq \frac{1}{4\varepsilon^2 \alpha}.$$

Thus, choosing

$$n \geq \frac{1}{4\varepsilon^2 \alpha}$$

guarantees

$$P(|\widehat{p} - p| < \varepsilon) \geq 1 - \alpha$$

for every Bernoulli success probability p.

Problem 56

A quality-control officer wants to estimate the proportion p of defective nails produced by a company.

A random sample of n nails is inspected, and

$$\widehat{p}$$

denotes the fraction of defective nails in the sample.

Using Chebyshev's inequality, determine a sample size sufficient to guarantee that, with probability at least 98%,

$$|\widehat{p} - p| < 0.03,$$

regardless of the unknown value of p.

Solution

From the general Bernoulli estimation bound,

$$n \geq \frac{1}{4\varepsilon^2\alpha}$$

is sufficient to guarantee

$$P(|\widehat{p} - p| < \varepsilon) \geq 1 - \alpha.$$

Here,

$$\varepsilon = 0.03$$

and

$$1 - \alpha = 0.98,$$

so

$$\alpha = 0.02.$$

Therefore,

$$\begin{aligned} n &\geq \frac{1}{4(0.03)^2(0.02)} \\ &= 13888.888\ldots. \end{aligned}$$

Since the sample size must be an integer,

$$n \geq 13889.$$

Thus, at least

$$13{,}889$$

nails should be inspected.

Problem 57

A coin has an unknown probability p of landing heads.

The coin is flipped independently 3000 times, and

$$\widehat{p}$$

denotes the fraction of flips that result in heads.

Use Chebyshev's inequality to show that

$$P(|\widehat{p} - p| < 0.03) \geq 0.90.$$

Solution

Let

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th flip is heads,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$X_i \sim \text{Bernoulli}(p)$$

and

$$\widehat{p} = \frac{1}{3000} \sum_{i=1}^{3000} X_i.$$

Therefore,

$$E(\widehat{p}) = p$$

and

$$\text{Var}(\widehat{p}) = \frac{p(1-p)}{3000}.$$

By Chebyshev's inequality,

$$P(|\widehat{p} - p| \geq 0.03) \leq \frac{p(1-p)}{(0.03)^2(3000)}.$$

Since

$$p(1-p) \leq \frac{1}{4},$$

we obtain

$$P(|\widehat{p} - p| \geq 0.03) \leq \frac{1}{4(0.03)^2(3000)}.$$

Thus,

$$\begin{aligned} P(|\widehat{p} - p| < 0.03) &\geq 1 - \frac{1}{4(0.03)^2(3000)} \\ &= 1 - \frac{1}{10.8} \\ &\approx 0.9074. \end{aligned}$$

Therefore,

$$P(|\widehat{p} - p| < 0.03) \geq 0.9074 > 0.90.$$

Hence,

$$P(|\widehat{p} - p| < 0.03) \geq 0.90.$$

Problem 58

Let X be a random variable and let $k > 0$ be a constant. Prove that

$$P(X > t) \leq \frac{E(e^{kX})}{e^{kt}},$$

whenever $E(e^{kX}) < \infty$.

Solution

Since the exponential function is increasing and $k > 0$,

$$X > t$$

implies

$$e^{kX} > e^{kt}.$$

Therefore,

$$P(X > t) = P(e^{kX} > e^{kt}).$$

The random variable

$$e^{kX}$$

is nonnegative. Hence, by Markov's inequality,

$$P(e^{kX} \geq e^{kt}) \leq \frac{E(e^{kX})}{e^{kt}}.$$

Since

$$\{e^{kX} > e^{kt}\} \subseteq \{e^{kX} \geq e^{kt}\},$$

we obtain

$$P(X > t) \leq \frac{E(e^{kX})}{e^{kt}}.$$

Problem 59

Suppose that random variables X and Y satisfy

$$E[(X-Y)^2] = 0.$$

Prove that

$$P(X = Y) = 1.$$

Solution

The random variable

$$(X-Y)^2$$

is nonnegative.
For every $\varepsilon > 0$, Markov's inequality gives

$$P\left((X-Y)^2 \geq \varepsilon\right) \leq \frac{E[(X-Y)^2]}{\varepsilon}.$$

Since

$$E[(X-Y)^2] = 0,$$

we obtain

$$P\left((X-Y)^2 \geq \varepsilon\right) = 0$$

for every $\varepsilon > 0$.
Now

$$\{X \neq Y\} = \{(X-Y)^2 > 0\}.$$

Also,

$$\{(X-Y)^2 > 0\} = \bigcup_{n=1}^{\infty} \left\{(X-Y)^2 \geq \frac{1}{n}\right\}.$$

Therefore,

$$\begin{aligned} P(X \neq Y) &\leq \sum_{n=1}^{\infty} P\left((X-Y)^2 \geq \frac{1}{n}\right) \\ &= 0. \end{aligned}$$

Hence,

$$P(X \neq Y) = 0,$$

and therefore

$$P(X = Y) = 1.$$

Problem 60

Let

$$X_1, X_2, \ldots$$

be independent and identically distributed nonnegative random variables with density

$$f(x) = \begin{cases} 4x(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n}.$$

Solution

As written, the proposed function is not a probability density function, because

$$\begin{aligned} \int_0^1 4x(1-x)\,dx &= 4\int_0^1 (x - x^2)\,dx \\ &= 4\left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{2}{3} \ne 1. \end{aligned}$$

Therefore, the problem as stated does not define a valid distribution.
If the intended normalized density is

$$f(x) = 6x(1-x), \qquad 0 \le x \le 1,$$

then

$$\begin{aligned} E(X_i) &= \int_0^1 x\,6x(1-x)\,dx \\ &= 6\int_0^1 (x^2 - x^3)\,dx \\ &= 6\left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{1}{2}. \end{aligned}$$

By the Strong Law of Large Numbers,

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow E(X_1)$$

with probability 1.
Thus, under the normalized density,

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow \frac{1}{2} \qquad \text{almost surely.}$$

Problem 61

The time X, measured in hours, required for a student to finish an aptitude test has probability density function

$$f(x) = \begin{cases} 6(x-1)(2-x), & 1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

A random sample of 15 students is selected.

Using the Central Limit Theorem, approximate the probability that the average time required to complete the test is less than 1 hour and 25 minutes.

Solution

Let

$$X_1, \dots, X_{15}$$

be the completion times and let

$$\overline{X} = \frac{1}{15}\sum_{i=1}^{15} X_i.$$

First calculate the population mean:

$$\begin{aligned} E(X) &= \int_1^2 x\,6(x-1)(2-x)\,dx \\ &= \frac{3}{2}. \end{aligned}$$

Also,

$$\begin{aligned} E(X^2) &= \int_1^2 x^2\,6(x-1)(2-x)\,dx \\ &= \frac{23}{10}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{23}{10} - \frac{9}{4} \\ &= \frac{1}{20}. \end{aligned}$$

Thus,

$$\mu = \frac{3}{2}, \qquad \sigma^2 = \frac{1}{20}.$$

By the Central Limit Theorem,

$$\overline{X} \approx N\left(\frac{3}{2}, \frac{1}{20(15)}\right) = N\left(\frac{3}{2}, \frac{1}{300}\right).$$

Now

$$1 \text{ hour } 25 \text{ minutes} = 1 + \frac{25}{60} = \frac{17}{12}$$

hours.

Therefore,

$$P\left(\overline{X} < \frac{17}{12}\right) \approx P\left(Z < \frac{\frac{17}{12} - \frac{3}{2}}{1/\sqrt{300}}\right)$$
$$= P(Z < -1.443).$$

Hence,

$$P\left(\overline{X} < \frac{17}{12}\right) \approx \Phi(-1.443) \approx 0.075.$$

Thus, the approximate probability is

$$0.075.$$

Problem 62

Twenty numbers are selected independently and uniformly from the interval $(0, 1)$.

Using the Central Limit Theorem, approximate the probability that their sum is at least 8.

Solution

Let

$$X_1, \ldots, X_{20} \overset{\text{i.i.d.}}{\sim} \text{Unif}(0, 1)$$

and define

$$S_{20} = \sum_{i=1}^{20} X_i.$$

For a $\text{Unif}(0, 1)$ random variable,

$$E(X_i) = \frac{1}{2}$$

and

$$\text{Var}(X_i) = \frac{1}{12}.$$

Therefore,

$$E(S_{20}) = 20\left(\frac{1}{2}\right) = 10,$$

and

$$\text{Var}(S_{20}) = 20\left(\frac{1}{12}\right) = \frac{5}{3}.$$

By the Central Limit Theorem,

$$S_{20} \approx N\left(10, \frac{5}{3}\right).$$

Hence,

$$\begin{aligned} P(S_{20} \geq 8) &\approx P\left(Z \geq \frac{8 - 10}{\sqrt{5/3}}\right) \\ &= P(Z \geq -1.549). \end{aligned}$$

Therefore,

$$\begin{aligned} P(S_{20} \geq 8) &\approx 1 - \Phi(-1.549) \\ &= \Phi(1.549) \\ &\approx 0.939. \end{aligned}$$

Thus, the approximate probability is

$$0.939.$$

Problem 63

A biologist wants to estimate the mean lifetime ℓ of a certain type of insect.

Suppose the insect lifetimes are independent random variables with

$$E(X_i) = \ell$$

and

$$\mathrm{Var}(X_i) = 1.5$$

days squared.

The biologist estimates ℓ using the sample mean

$$\overline{X}.$$

Using the Central Limit Theorem, determine approximately how large the sample size n should be so that

$$P(|\overline{X} - \ell| < 0.2) \approx 0.98.$$

Solution

For a sample of size n, $E(\overline{X}) = \ell$ and $\mathrm{Var}(\overline{X}) = \frac{1.5}{n}$. By the Central Limit Theorem,

$$\frac{\overline{X} - \ell}{\sqrt{1.5/n}} \approx N(0,1).$$

Thus,

$$P(|\overline{X} - \ell| < 0.2) \approx P\left(|Z| < \frac{0.2\sqrt{n}}{\sqrt{1.5}}\right).$$

We want this probability to be approximately 0.98. Therefore,

$$P(-z < Z < z) = 0.98.$$

This requires

$$\Phi(z) = 0.99.$$

Using

$$z_{0.99} \approx 2.33,$$

we set

$$\frac{0.2\sqrt{n}}{\sqrt{1.5}} \approx 2.33.$$

Hence,

$$\sqrt{n} \approx \frac{2.33\sqrt{1.5}}{0.2}.$$

Squaring,

$$n \approx \left(\frac{2.33\sqrt{1.5}}{0.2}\right)^2 \approx 203.58.$$

Therefore, the sample size should be rounded up to

$$n = 204.$$

Problem 64

A random sample of size 24 is taken from a distribution with probability density function

$$f(x) = \begin{cases} \frac{1}{9}\left(x + \frac{5}{2}\right), & 1 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\overline{X}$ be the sample mean.

Using the Central Limit Theorem, approximate

$$P(2 < \overline{X} < 2.15).$$

Solution

First calculate the population mean:

$$\begin{aligned} \mu &= E(X) \\ &= \int_1^3 x\frac{1}{9}\left(x + \frac{5}{2}\right)\,dx \\ &= \frac{56}{27}. \end{aligned}$$

Next,

$$\begin{aligned} E(X^2) &= \int_1^3 x^2\frac{1}{9}\left(x + \frac{5}{2}\right)\,dx \\ &= \frac{125}{27}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma^2 &= E(X^2) - [E(X)]^2 \\ &= \frac{125}{27} - \left(\frac{56}{27}\right)^2 \\ &= \frac{239}{729}. \end{aligned}$$

For $n = 24$,

$$E(\overline{X}) = \frac{56}{27},$$

and

$$\operatorname{Var}(\overline{X}) = \frac{239}{729(24)}.$$

Thus,

$$\operatorname{SD}(\overline{X}) = \sqrt{\frac{239}{729(24)}}.$$

By the Central Limit Theorem,

$$\overline{X} \approx N\left(\frac{56}{27}, \frac{239}{729(24)}\right).$$

Therefore,

$$\begin{aligned} P(2 < \overline{X} < 2.15) &\approx P\left(\frac{2 - \frac{56}{27}}{\sqrt{239/[729(24)]}} < Z < \frac{2.15 - \frac{56}{27}}{\sqrt{239/[729(24)]}}\right) \\ &= P(-0.634 < Z < 0.650). \end{aligned}$$

Hence,

$$P(2 < \overline{X} < 2.15) \approx \Phi(0.650) - \Phi(-0.634)$$
$$\approx 0.479.$$

Thus, the approximate probability is

$$0.479.$$

Problem 65

A random sample of size $n \geq 1$ is taken from the distribution with probability density function

$$f(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty.$$

Let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Find

$$P(\overline{X} > 0).$$

Solution

The density satisfies

$$f(x) = f(-x),$$

so the distribution of each X_i is symmetric about 0.
Since the X_i's are independent, their sum

$$S_n = X_1 + \cdots + X_n$$

is also symmetric about 0.
Because the distribution is continuous,

$$P(S_n = 0) = 0.$$

By symmetry,

$$P(S_n > 0) = P(S_n < 0).$$

These two probabilities sum to 1, so

$$P(S_n > 0) = \frac{1}{2}.$$

Since

$$\overline{X} > 0$$

if and only if

$$S_n > 0,$$

we obtain

$$P(\overline{X} > 0) = \frac{1}{2}.$$

Problem 66

For the scores on an achievement test given to a certain population of students, the population mean is

$$\mu = 500$$

and the population standard deviation is

$$\sigma = 100.$$

Let $\overline{X}$ be the mean of the scores of a random sample of 35 students.

Using the Central Limit Theorem, estimate

$$P(460 < \overline{X} < 540).$$

Solution

For a sample of size

$$n = 35,$$

the sample mean satisfies

$$E(\overline{X}) = 500$$

and

$$\mathrm{SD}(\overline{X}) = \frac{100}{\sqrt{35}}.$$

By the Central Limit Theorem,

$$\overline{X} \approx N\left(500, \frac{100^2}{35}\right).$$

Therefore,

$$\begin{aligned} P(460 < \overline{X} < 540) &\approx P\left(\frac{460-500}{100/\sqrt{35}} < Z < \frac{540-500}{100/\sqrt{35}}\right) \\ &= P(-2.366 < Z < 2.366). \end{aligned}$$

Hence,

$$\begin{aligned} P(460 < \overline{X} < 540) &\approx \Phi(2.366) - \Phi(-2.366) \\ &= 2\Phi(2.366) - 1 \\ &\approx 0.982. \end{aligned}$$

Thus, the approximate probability is

$$0.982.$$

Problem 67

Let

$$X \sim \mathrm{Unif}(0,1),$$

and define

$$Y = -\ln X.$$

(a) Find the cumulative distribution function $F_Y(y)$.

(b) Find the probability density function $f_Y(y)$.

(c) Identify the distribution of Y.

Solution

Since

$$0 < X < 1,$$

we have

$$-\ln X > 0.$$

Therefore,

$$Y > 0.$$

(a) For $y \le 0$,

$$F_Y(y) = 0.$$

Now suppose $y > 0$. Then

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(-\ln X \le y). \end{aligned}$$

Since

$$-\ln X \le y$$

is equivalent to

$$\ln X \ge -y,$$

and hence

$$X \ge e^{-y},$$

we obtain

$$\begin{aligned} F_Y(y) &= P(X \ge e^{-y}) \\ &= 1 - P(X < e^{-y}). \end{aligned}$$

Because

$$X \sim \mathrm{Unif}(0,1),$$

we have

$$P(X < e^{-y}) = e^{-y}.$$

Therefore,

$$F_Y(y) = 1 - e^{-y}, \qquad y > 0.$$

Thus,

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ 1 - e^{-y}, & y > 0. \end{cases}$$

(b) Differentiating the CDF,

$$f_Y(y) = F_Y'(y).$$

Hence,

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) This is the density of an exponential random variable with parameter 1. Therefore,

$$Y \sim \text{Exp}(1).$$

Problem 68

Let

$$X \sim N(0,1),$$

and define

$$Y = X^2.$$

(a) Explain why the transformation

$$g(x) = x^2$$

is not one-to-one.

(b) Find the cumulative distribution function $F_Y(y)$.

(c) Find the probability density function $f_Y(y)$.

Solution

(a) The transformation

$$g(x) = x^2$$

is not one-to-one because, for every $x \neq 0$,

$$g(x) = g(-x).$$

Thus, a given positive value of Y has two possible preimages:

$$x = \sqrt{y} \qquad \text{and} \qquad x = -\sqrt{y}.$$

(b) Since

$$Y = X^2,$$

we have

$$Y \geq 0.$$

Therefore,

$$F_Y(y) = 0, \qquad y < 0.$$

Now suppose $y \geq 0$. Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y). \end{aligned}$$

The event

$$X^2 \leq y$$

is equivalent to

$$-\sqrt{y} \leq X \leq \sqrt{y}.$$

Hence,

$$\begin{aligned} F_Y(y) &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}). \end{aligned}$$

Using

$$\Phi(-a) = 1 - \Phi(a),$$

we obtain

$$F_Y(y) = 2\Phi(\sqrt{y}) - 1.$$

Thus,

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 2\Phi(\sqrt{y}) - 1, & y \geq 0. \end{cases}$$

(c) For $y > 0$,

$$f_Y(y) = \frac{d}{dy}\left[2\Phi(\sqrt{y}) - 1\right].$$

Using the chain rule,

$$\begin{aligned} f_Y(y) &= 2\phi(\sqrt{y})\frac{1}{2\sqrt{y}} \\ &= \frac{\phi(\sqrt{y})}{\sqrt{y}}. \end{aligned}$$

Since

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

we have

$$\phi(\sqrt{y}) = \frac{1}{\sqrt{2\pi}}e^{-y/2}.$$

Therefore,

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}}e^{-y/2}, \qquad y > 0.$$

Thus,

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}}e^{-y/2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 69

Let X and Y be independent exponential random variables with common parameter $\lambda > 0$:

$$X, Y \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda).$$

Define

$$T = X + Y.$$

Using the convolution formula

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx,$$

find the probability density function of T.

Be sure to determine the correct limits of integration and the support of T.

Solution

Since

$$X, Y \sim \text{Exp}(\lambda),$$

their common density is

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $X > 0$ and $Y > 0$,

$$T = X + Y > 0.$$

Therefore,

$$f_T(t) = 0, \qquad t \le 0.$$

Now suppose $t > 0$. The convolution formula gives

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx.$$

For the integrand to be nonzero, we need both

$$x > 0$$

and

$$t - x > 0.$$

Thus,

$$0 < x < t.$$

Therefore,

$$\begin{aligned} f_T(t) &= \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}\, dx \\ &= \lambda^2 e^{-\lambda t} \int_0^t dx \\ &= \lambda^2 t e^{-\lambda t}. \end{aligned}$$

Hence,

$$f_T(t) = \begin{cases} \lambda^2 t e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 70

Let

$$X, Y \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(0,1),$$

with X and Y independent, and define

$$T = X + Y.$$

Using the convolution formula

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx,$$

find the probability density function of T.

Solution

Since

$$X, Y \sim \mathrm{Unif}(0,1),$$

we have

$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$0 < X < 1 \qquad \text{and} \qquad 0 < Y < 1,$$

so

$$0 < T < 2.$$

Therefore,

$$f_T(t) = 0$$

outside the interval $(0,2)$.
For $0 < t \le 1$, the convolution formula gives

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx.$$

For the integrand to be nonzero, we need

$$0 < x < 1$$

and

$$0 < t - x < 1.$$

When $0 < t \le 1$, these conditions reduce to

$$0 < x < t.$$

Therefore,

$$f_T(t) = \int_0^t 1\,dx = t.$$

Now suppose

$$1 < t < 2.$$

Again, we need

$$0 < x < 1$$

and

$$0 < t - x < 1.$$

The second inequality gives

$$t - 1 < x < t.$$

Combining this with $0 < x < 1$, we obtain

$$t - 1 < x < 1.$$

Hence,

$$f_T(t) = \int_{t-1}^{1} 1\,dx = 2 - t.$$

Therefore,

$$f_T(t) = \begin{cases} t, & 0 < t \leq 1, \\ 2 - t, & 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 71

Let X and Y be independent random variables satisfying

$$P(X = 1) = P(X = -1) = \frac{1}{2},$$

and

$$P(Y = 1) = P(Y = -1) = \frac{1}{2}.$$

Define

$$Z = XY.$$

(a) Find the probability mass function of Z.

(b) Show that X and Z are independent.

(c) Show that Y and Z are independent.

(d) Conclude that X, Y, Z are pairwise independent.

(e) Show that X, Y, Z are not mutually independent.

Solution

The four possible values of (X, Y) are

$$(1, 1), \quad (1, -1), \quad (-1, 1), \quad (-1, -1),$$

each with probability 1/4.
Since

$$Z = XY,$$

we obtain the table

X	Y	Z
1	1	1
1	−1	−1
−1	1	−1
−1	−1	1

(a) We have

$$P(Z = 1) = P(X = Y) = \frac{1}{2},$$

and

$$P(Z = -1) = P(X \neq Y) = \frac{1}{2}.$$

Thus,

$$p_Z(z) = \begin{cases} \frac{1}{2}, & z = 1, -1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) To show that X and Z are independent, consider their joint probabilities.

For example,

$$P(X = 1, Z = 1) = P(X = 1, Y = 1) = \frac{1}{4}.$$

Also,

$$P(X=1)P(Z=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Similarly,

$$P(X=1, Z=-1) = P(X=1, Y=-1) = \frac{1}{4},$$

and

$$P(X=1)P(Z=-1) = \frac{1}{4}.$$

The same calculation holds for $X = -1$. Therefore,

$$P(X=x, Z=z) = P(X=x)P(Z=z)$$

for every

$$x, z \in \{-1, 1\}.$$

Hence, X and Z are independent.

(c) By symmetry, the same argument shows that

$$Y$$

and

$$Z$$

are independent.

Explicitly,

$$P(Y=y, Z=z) = P(Y=y)P(Z=z)$$

for all

$$y, z \in \{-1, 1\}.$$

(d) We are given that X and Y are independent. We have also shown that X and Z are independent and that Y and Z are independent.

Therefore, X, Y, Z are pairwise independent.

(e) However,

$$XYZ = XY(XY) = X^2Y^2 = 1$$

with probability 1.

Thus, the three random variables satisfy a deterministic relationship.

For example,

$$P(X=1, Y=1, Z=-1) = 0.$$

But

$$P(X=1)P(Y=1)P(Z=-1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Therefore,

$$P(X=1, Y=1, Z=-1) \neq P(X=1)P(Y=1)P(Z=-1).$$

Hence, X, Y, Z are not mutually independent.

Problem 72

Let X and Y have joint probability density function

$$f(x,y) = \begin{cases} 2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the joint cumulative distribution function

$$F_{X,Y}(x,y) = P(X \le x, Y \le y).$$

(b) Use the joint CDF to find the marginal CDFs

$$F_X(x) \qquad \text{and} \qquad F_Y(y).$$

(c) Use the joint CDF to calculate

$$P\left(\frac{1}{4} < X \le \frac{3}{4}, \frac{1}{4} < Y \le \frac{1}{2}\right).$$

Solution

The support is the triangular region

$$0 < y < x < 1.$$

(a) We calculate

$$F_{X,Y}(x,y) = P(X \le x, Y \le y).$$

If

$$x \le 0 \qquad \text{or} \qquad y \le 0,$$

then

$$F_{X,Y}(x,y) = 0.$$

Now consider

$$0 < x < 1, \qquad 0 < y < x.$$

For a fixed value v of Y, the support requires

$$v < u < x,$$

where u denotes the value of X.

Thus,

$$\begin{aligned} F_{X,Y}(x,y) &= \int_0^y \int_v^x 2\,du\,dv \\ &= 2\int_0^y (x-v)\,dv \\ &= 2xy - y^2. \end{aligned}$$

Next consider

$$0 < x < 1, \qquad y \ge x.$$

In this case, the condition $Y \le y$ does not further restrict the support below $X = x$. Therefore,

$$\begin{aligned} F_{X,Y}(x,y) &= P(X \le x) \\ &= \int_0^x \int_0^u 2\,dv\,du \\ &= \int_0^x 2u\,du \\ &= x^2. \end{aligned}$$

Now suppose

$$x \ge 1.$$

If

$$0 < y < 1,$$

then the condition $X \le x$ imposes no restriction, so

$$\begin{aligned} F_{X,Y}(x,y) &= P(Y \le y) \\ &= \int_0^y \int_v^1 2\,du\,dv \\ &= 2\int_0^y (1-v)\,dv \\ &= 2y - y^2. \end{aligned}$$

Finally, if

$$x \ge 1 \qquad \text{and} \qquad y \ge 1,$$

then

$$F_{X,Y}(x,y) = 1.$$

Thus,

$$F_{X,Y}(x,y) = \begin{cases} 0, & x \le 0 \text{ or } y \le 0, \\ 2xy - y^2, & 0 < y < x < 1, \\ x^2, & 0 < x < 1,\ y \ge x, \\ 2y - y^2, & x \ge 1,\ 0 < y < 1, \\ 1, & x \ge 1,\ y \ge 1. \end{cases}$$

(b) The marginal CDF of X is

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y).$$

Hence,

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ x^2, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

Similarly,

$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y).$$

Thus,

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ 2y - y^2, & 0 < y < 1, \\ 1, & y \ge 1. \end{cases}$$

(c) For a joint CDF,

$$\begin{aligned} &P(a < X \le b,\ c < Y \le d) \\ &= F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c). \end{aligned}$$

Here,

$$a = \frac{1}{4}, \qquad b = \frac{3}{4}, \qquad c = \frac{1}{4}, \qquad d = \frac{1}{2}.$$

First,

$$F_{X,Y}\left(\frac{3}{4}, \frac{1}{2}\right) = 2\left(\frac{3}{4}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

Next, since

$$\frac{1}{2} \ge \frac{1}{4},$$

we use the $y \ge x$ case:

$$F_{X,Y}\left(\frac{1}{4}, \frac{1}{2}\right) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Also,

$$F_{X,Y}\left(\frac{3}{4}, \frac{1}{4}\right) = 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^2 = \frac{5}{16}.$$

Finally,

$$F_{X,Y}\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Therefore,

$$\begin{aligned} &P\left(\frac{1}{4} < X \le \frac{3}{4}, \frac{1}{4} < Y \le \frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{16} - \frac{5}{16} + \frac{1}{16} \\ &= \frac{3}{16}. \end{aligned}$$

Thus,

$$P\left(\frac{1}{4} < X \le \frac{3}{4}, \frac{1}{4} < Y \le \frac{1}{2}\right) = \frac{3}{16}.$$

References

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