University of Washington
Department of Mathematics

Final Examination Solutions

MATH 394: Probability I

Summer 2026 — Instructor: Arman Jahangiri

Exam date: August 21, 2026
Answering time: 60 minutes
Total: 60 points

Problem 1. **16 points**

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} x^2 e^{-x(y+1)}, & x \geq 0, \quad y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

The marginal probability density functions are

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{2}{(1+y)^3}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate

$$\text{Cov}(X, Y).$$

(4 points)

(b) Calculate

$$\text{Cov}(3X + 1,\, 2X - 4).$$

(3 points)

(c) Are X and Y independent? Justify your answer. **(2 points)**

(d) Calculate the moment-generating function of X, namely

$$M_X(t) = E(e^{tX}).$$

(3 points)

(e) Using $M_X(t)$, calculate $E(X)$. **(2 points)**

Solution

(a) Recall that

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

First,

$$\begin{aligned} E(X) &= \int_0^\infty x f_X(x)\, dx \\ &= \int_0^\infty x^2 e^{-x}\, dx \\ &= 2. \end{aligned}$$

Next,

$$\begin{aligned} E(Y) &= \int_0^\infty y \frac{2}{(1+y)^3}\, dy \\ &= 1. \end{aligned}$$

To calculate $E(XY)$, use the joint density:

$$\begin{aligned} E(XY) &= \int_0^\infty \int_0^\infty xy\, x^2 e^{-x(y+1)}\, dy\, dx \\ &= \int_0^\infty x^3 e^{-x} \left(\int_0^\infty y e^{-xy}\, dy \right) dx. \end{aligned}$$

For $x > 0$,

$$\int_0^\infty y e^{-xy}\, dy = \frac{1}{x^2}.$$

Therefore,

$$\begin{aligned} E(XY) &= \int_0^\infty x e^{-x}\, dx \\ &= 1. \end{aligned}$$

Hence

$$\mathrm{Cov}(X, Y) = 1 - (2)(1) = -1.$$

(b) Using

$$\mathrm{Cov}(aX + b, cX + d) = ac\, \mathrm{Var}(X),$$

we obtain

$$\mathrm{Cov}(3X + 1, 2X - 4) = 6\, \mathrm{Var}(X).$$

Now

$$E(X) = 2,$$

and

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 f_X(x)\, dx \\ &= \int_0^\infty x^3 e^{-x}\, dx \\ &= 6. \end{aligned}$$

Thus

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

Therefore,

$$\mathrm{Cov}(3X + 1, 2X - 4) = 6(2) = 12.$$

(c) If X and Y were independent, then we would have

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

on their support.

However,

$$f_X(x)f_Y(y) = xe^{-x}\frac{2}{(1+y)^3},$$

whereas

$$f_{X,Y}(x,y) = x^2e^{-x(y+1)}.$$

These are not equal in general. Therefore, X and Y are not independent.
Alternatively, part (a) already gives

$$\mathrm{Cov}(X,Y) = -1 \neq 0.$$

Since independence would imply zero covariance, X and Y cannot be independent.
(d) By definition,

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx}xe^{-x}\,dx \\ &= \int_0^\infty xe^{-(1-t)x}\,dx. \end{aligned}$$

For $t < 1$,

$$\int_0^\infty xe^{-(1-t)x}\,dx = \frac{1}{(1-t)^2}.$$

Therefore,

$$M_X(t) = \frac{1}{(1-t)^2}, \qquad t < 1.$$

(e) We know that

$$E(X) = M_X'(0).$$

Since

$$M_X(t) = (1-t)^{-2},$$

we have

$$M_X'(t) = 2(1-t)^{-3}.$$

Therefore,

$$E(X) = M_X'(0) = 2.$$

Final Answer

$$\text{(a)} \quad \mathrm{Cov}(X,Y) = -1.$$

$$\text{(b)} \quad \mathrm{Cov}(3X+1, 2X-4) = 12.$$

$$\text{(c)} \quad X \text{ and } Y \text{ are not independent.}$$

$$\text{(d)} \quad M_X(t) = \frac{1}{(1-t)^2}, \qquad t < 1.$$

$$\text{(e)} \quad E(X) = 2.$$

Problem 2. **8 points**

Let X and Y be two positive independent continuous random variables with probability density functions $f_1(x)$ and $f_2(y)$, respectively.

Find the probability density function of

$$U = \frac{X}{Y}.$$

Hint: Let $V = X$. Find the joint probability density function of U and V (Jacobian method), and then calculate the marginal probability density function of U.

Solution

Define

$$U = \frac{X}{Y}, \qquad V = X.$$

We first find the inverse transformation.
Since

$$v = x$$

and

$$u = \frac{x}{y},$$

we obtain

$$x = v, \qquad y = \frac{v}{u}.$$

Because $X > 0$ and $Y > 0$, the support of (U, V) is

$$u > 0, \qquad v > 0.$$

The Jacobian of the inverse transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Since

$$x = v, \qquad y = \frac{v}{u},$$

we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u^2}.$$

Thus

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{v}{u^2}.$$

Since X and Y are independent,

$$f_{X,Y}(x,y) = f_1(x) f_2(y).$$

Therefore,

$$f_{U,V}(u,v) = f_1(v) f_2\left(\frac{v}{u}\right) \frac{v}{u^2}, \qquad u, v > 0.$$

Finally, integrate out V:

$$f_U(u) = \int_0^\infty f_1(v) f_2\left(\frac{v}{u}\right) \frac{v}{u^2}\, dv, \qquad u > 0.$$

For $u \leq 0$,

$$f_U(u) = 0.$$

Final Answer

$$f_U(u) = \begin{cases} \dfrac{1}{u^2} \displaystyle\int_0^\infty v f_1(v) f_2\left(\frac{v}{u}\right)\, dv, & u > 0, \\ 0, & u \leq 0. \end{cases}$$

Problem 3. **7 points**

Let X and Y be i.i.d. positive random variables.

For each part below, fill in the appropriate equality or inequality symbol.

Write $=$ if the two sides are always equal, $\leq$ if the left-hand side is less than or equal to the right-hand side but not necessarily equal, and similarly for $\geq$. If no relation holds in general, write ?.

Justify each answer.

(a)

$$E(\log X) \quad \_\_\_\_\_\_ \quad \log(E(X)).$$

(3 points)

(b)

$$P(X \leq Y) \quad \_\_\_\_\_\_ \quad P(X \geq Y).$$

(2 points)

(c)

$$E(XY) \quad \_\_\_\_\_\_ \quad \sqrt{E(X^2)E(Y^2)}.$$

(2 points)

Solution

(a) The function

$$g(x) = \log x$$

is concave on $(0, \infty)$.
By Jensen's inequality for a concave function,

$$E[g(X)] \leq g(E[X]).$$

Therefore,

$$E(\log X) \leq \log(E(X)).$$

(b) Since X and Y are i.i.d., the joint distribution of (X, Y) is symmetric under interchange of X and Y.
Therefore,

$$P(X < Y) = P(Y < X).$$

Also,

$$P(X \leq Y) = P(X < Y) + P(X = Y)$$

and

$$P(X \geq Y) = P(X > Y) + P(X = Y).$$

Since

$$P(X < Y) = P(X > Y),$$

it follows that

$$P(X \leq Y) = P(X \geq Y).$$

(c) By the Cauchy–Schwarz inequality,

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}.$$

Since X and Y are positive,

$$E(XY) \geq 0.$$

Hence

$$E(XY) \leq \sqrt{E(X^2)E(Y^2)}.$$

In fact, because X and Y are independent,

$$E(XY) = E(X)E(Y),$$

but Cauchy–Schwarz gives the requested comparison directly.

Final Answer

$$\text{(a)} \quad E(\log X) \leq \log(E(X)).$$

$$\text{(b)} \quad P(X \leq Y) = P(X \geq Y).$$

$$\text{(c)} \quad E(XY) \leq \sqrt{E(X^2)E(Y^2)}.$$

Problem 4. **7 points**

Suppose

$$X_1, \ldots, X_n$$

are independent Bernoulli random variables with unknown success probability p, and define

$$\widehat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Let

$$\varepsilon > 0 \qquad \text{and} \qquad 0 < \alpha < 1.$$

Using Chebyshev's inequality, derive a condition on the sample size n that guarantees

$$P\left(|\widehat{p} - p| < \varepsilon\right) \geq 1 - \alpha$$

for every

$$0 < p < 1.$$

Solution

Since

$$X_i \sim \text{Bernoulli}(p),$$

we have

$$E(X_i) = p, \qquad \text{Var}(X_i) = p(1-p).$$

Therefore,

$$E(\widehat{p}) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = p.$$

Since the X_i's are independent,

$$\begin{aligned} \text{Var}(\widehat{p}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i) \\ &= \frac{p(1-p)}{n}. \end{aligned}$$

Chebyshev's inequality gives

$$P\left(|\widehat{p} - p| \geq \varepsilon\right) \leq \frac{\text{Var}(\widehat{p})}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2}.$$

We want a bound that holds for every $0 < p < 1$.
Since

$$p(1-p) \leq \frac{1}{4},$$

we obtain

$$P\left(|\widehat{p} - p| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2}.$$

Therefore,

$$P\left(|\widehat{p} - p| < \varepsilon\right) \geq 1 - \frac{1}{4n\varepsilon^2}.$$

To guarantee that this is at least $1 - \alpha$, it is sufficient that

$$\frac{1}{4n\varepsilon^2} \leq \alpha.$$

Solving for n,

$$n \geq \frac{1}{4\alpha\varepsilon^2}.$$

Since n must be an integer, one may take

$$n \geq \left\lceil \frac{1}{4\alpha\varepsilon^2} \right\rceil.$$

Final Answer

A sufficient condition is

$$n \geq \frac{1}{4\alpha\varepsilon^2}.$$

Equivalently, for integer sample size,

$$n \geq \left\lceil \frac{1}{4\alpha\varepsilon^2} \right\rceil.$$

Problem 5. **8 points**

Let X and Y be independent exponential random variables with common parameter $\lambda > 0$:

$$X, Y \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda),$$

with probability density function

$$f(w) = \begin{cases} \lambda e^{-\lambda w}, & w > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$T = X + Y.$$

Using the convolution formula

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx,$$

find the probability density function of T, including its support.

Solution

Since

$$T = X + Y$$

and $X, Y > 0$, we must have

$$T > 0.$$

Thus, for $t \le 0$,

$$f_T(t) = 0.$$

Now suppose $t > 0$.
For the integrand

$$f_X(x) f_Y(t-x)$$

to be nonzero, we need

$$x > 0$$

and

$$t - x > 0.$$

Therefore,

$$0 < x < t.$$

Using convolution,

$$\begin{aligned} f_T(t) &= \int_0^t f_X(x) f_Y(t-x)\, dx \\ &= \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}\, dx \\ &= \lambda^2 e^{-\lambda t} \int_0^t dx \\ &= \lambda^2 t e^{-\lambda t}. \end{aligned}$$

Hence

$$f_T(t) = \begin{cases} \lambda^2 t e^{-\lambda t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Thus T does *not* have an exponential distribution. It has a Gamma distribution with shape parameter 2 and rate parameter λ.

Final Answer

$$f_T(t) = \begin{cases} \lambda^2 t e^{-\lambda t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Equivalently,

$$T \sim \text{Gamma}(2, \lambda)$$

under the shape-rate parameterization.

Problem 6. **8 points**

Let X, Y, Z be jointly continuous with joint probability density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} x^2 e^{-x(1+y+z)}, & x,y,z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The marginal probability density functions are

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{(1+y)^2}, & y > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad f_Z(z) = \begin{cases} \dfrac{1}{(1+z)^2}, & z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X, Y, Z pairwise independent? Justify your answer. **(5 points)**

(b) Are X, Y, Z mutually independent? Justify your answer. **(3 points)**

Solution

(a) To determine pairwise independence, we examine the joint density of each pair.
First, consider X and Y. Integrating out Z,

$$\begin{aligned} f_{X,Y}(x,y) &= \int_0^\infty x^2 e^{-x(1+y+z)}\,dz \\ &= x^2 e^{-x(1+y)} \int_0^\infty e^{-xz}\,dz \\ &= x^2 e^{-x(1+y)} \frac{1}{x} \\ &= x e^{-x(1+y)}, \qquad x,y > 0. \end{aligned}$$

On the other hand,

$$f_X(x) f_Y(y) = e^{-x} \frac{1}{(1+y)^2}.$$

These are not equal in general:

$$x e^{-x(1+y)} \neq \frac{e^{-x}}{(1+y)^2}.$$

Therefore, X and Y are not independent.
Consequently, X, Y, Z cannot be pairwise independent.
For completeness, the same phenomenon occurs for the other pairs.
By symmetry,

$$f_{X,Z}(x,z) = x e^{-x(1+z)},$$

which is not equal to

$$f_X(x) f_Z(z) = \frac{e^{-x}}{(1+z)^2}.$$

For Y and Z,

$$f_{Y,Z}(y,z) = \int_0^\infty x^2 e^{-x(1+y+z)}\,dx.$$

Using

$$\int_0^\infty x^2 e^{-ax}\,dx = \frac{2}{a^3}, \qquad a > 0,$$

we obtain

$$f_{Y,Z}(y,z) = \frac{2}{(1+y+z)^3}.$$

However,

$$f_Y(y)f_Z(z) = \frac{1}{(1+y)^2(1+z)^2},$$

which is again different in general.
Hence none of the three pairs is independent.
(b) Mutual independence would require

$$f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$$

for all points in the support.
The product of the marginals is

$$f_X(x)f_Y(y)f_Z(z) = \frac{e^{-x}}{(1+y)^2(1+z)^2}.$$

But the joint density is

$$f_{X,Y,Z}(x,y,z) = x^2 e^{-x(1+y+z)}.$$

These expressions are not equal in general.
Therefore, X, Y, Z are not mutually independent.
Also, since mutual independence implies pairwise independence and part (a) shows that they are not pairwise independent, mutual independence is impossible.

Final Answer

$$\text{(a)} \quad X, Y, Z \text{ are not pairwise independent.}$$

In fact, none of the three pairs is independent.

$$\text{(b)} \quad X, Y, Z \text{ are not mutually independent.}$$

Problem 7. **6 points**

For each part below, select the **correct statement** of the indicated theorem. Only one choice is correct in each part.

(a) **Weak Law of Large Numbers (WLLN).**

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean

$$E(X_i) = \mu$$

and finite variance

$$\mathrm{Var}(X_i) = \sigma^2.$$

Let

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Which of the following is the conclusion of the Weak Law of Large Numbers? **(2 points)**

(A) For every $\varepsilon > 0$,

$$P\left(|\overline{X}_n - \mu| > \varepsilon\right) = 0 \qquad \text{for every } n.$$

(B) For every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|\overline{X}_n - \mu| > \varepsilon\right) = 0.$$

(C)

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1.$$

(D) For every $z \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le z\right) = \Phi(z).$$

(b) **Strong Law of Large Numbers (SLLN).**

Let $X_1, X_2, \ldots$ be i.i.d. random variables satisfying the same assumptions as in the Weak Law of Large Numbers, with

$$E(X_i) = \mu$$

and finite variance

$$\mathrm{Var}(X_i) = \sigma^2.$$

Which of the following is the conclusion of the Strong Law of Large Numbers? **(2 points)**

(A)

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1.$$

(B) For every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|\overline{X}_n - \mu| > \varepsilon\right) = 0.$$

(C)

$$\lim_{n\to\infty} P(\overline{X}_n = \mu) = 1.$$

(D)

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

(c) **Central Limit Theorem (CLT).**

Let $X_1, X_2, \ldots$ be i.i.d. random variables with

$$E(X_i) = \mu, \qquad \text{Var}(X_i) = \sigma^2 < \infty.$$

Which of the following is the conclusion of the Central Limit Theorem? **(2 points)**

(A)

$$\overline{X}_n \xrightarrow{P} N(\mu, \sigma^2).$$

(B)

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1.$$

(C) For every $z \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le z\right) = \Phi(z).$$

(D) For every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|\overline{X}_n - \mu| > \varepsilon\right) = 0.$$

Solution

(a) Weak Law of Large Numbers.
The WLLN states that the sample mean converges to the population mean μ *in probability.* That is, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|\overline{X}_n - \mu| > \varepsilon\right) = 0.$$

Therefore, the correct answer is

B.

(b) Strong Law of Large Numbers.
The SLLN states that

$$\overline{X}_n \longrightarrow \mu$$

almost surely. Equivalently,

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1.$$

Therefore, the correct answer is

A.

(c) Central Limit Theorem.
The CLT states that the standardized sample mean converges in distribution to a standard normal random variable:

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

Equivalently, for every $z \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le z\right) = \Phi(z).$$

Therefore, the correct answer is

C.

Final Answer

$\boxed{\text{(a) B}}$ $\boxed{\text{(b) A}}$ $\boxed{\text{(c) C}}$