

University of Washington
Department of Mathematics

Midterm Examination Solutions

MATH 394: Probability I

Summer 2026 — Instructor: Arman Jahangiri

Exam date: July 22, 2026
Answering time: 60 minutes
Total: 60 points

Problem 1.**5 points**

A student organization has 7 mathematics graduate students, 5 statistics graduate students, and 6 undergraduate students. A committee of 5 students is selected uniformly at random from all committees of 5.

Find the probability that the committee contains exactly 2 mathematics graduate students, exactly 1 statistics graduate student, and exactly 2 undergraduate students.

Give an exact answer. A correct expression involving binomial coefficients is acceptable.

Solution

There are

$$\binom{18}{5}$$

possible committees of 5 students, and all are equally likely.

To form a committee of the required type, choose:

- 2 of the 7 mathematics graduate students,
- 1 of the 5 statistics graduate students, and
- 2 of the 6 undergraduate students.

Thus the number of favorable committees is

$$\binom{7}{2} \binom{5}{1} \binom{6}{2}.$$

Therefore,

$$P(\text{required composition}) = \frac{\binom{7}{2} \binom{5}{1} \binom{6}{2}}{\binom{18}{5}}.$$

Numerically,

$$\binom{7}{2} \binom{5}{1} \binom{6}{2} = 21 \cdot 5 \cdot 15 = 1575, \quad \binom{18}{5} = 8568,$$

so

$$\frac{1575}{8568} = \frac{525}{2856} = \frac{175}{952}.$$

Problem 2.**10 points**

A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase “free money” is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention “free money.” What is the probability that it is spam?

Solution

(a) Let S be the event that the email is spam, and let F be the event that the email contains the phrase “free money.” We are given

$$P(S) = 0.80, \quad P(F | S) = 0.10, \quad P(F | S^c) = 0.01.$$

Bayes' rule gives

$$P(S | F) = \frac{P(F | S)P(S)}{P(F | S)P(S) + P(F | S^c)P(S^c)}.$$

Substituting the given values,

$$P(S | F) = \frac{0.10 \cdot 0.80}{0.10 \cdot 0.80 + 0.01 \cdot 0.20} = \frac{0.08}{0.082} = \frac{40}{41}.$$

Problem 3.**10 points**

Let A and B be events.

- (a) Let S be a sample space and $A_1, \dots, A_n \in \mathcal{F}$ be events. Explain what it means for A_i 's to form a partition of S , **(3 points)**
- (b) Suppose $P(A) = 0.6$ and $P(B) = 0.5$. Can A and B be disjoint? **(3 points)**
- (c) Suppose that A is independent of itself. Use the definition of independence to conclude that $P(A) = 0$ or $P(A) = 1$. **(4 points)**

Solution

- (a) The events $A_1, \dots, A_n$ form a partition of the sample space S if they are pairwise disjoint,

$$A_i \cap A_j = \emptyset, \quad i \neq j,$$

and

$$\bigcup_{i=1}^n A_i = S.$$

- (b) If A and B were disjoint, then

$$1 \geq P(A \cup B) = P(A) + P(B) = 0.6 + 0.5 = 1.1,$$

which is impossible since probabilities cannot exceed 1. Hence A and B cannot be disjoint.

- (c) If A is independent of itself, then

$$P(A \cap A) = P(A)P(A).$$

Since $A \cap A = A$,

$$P(A) = P(A)^2.$$

Let $p = P(A)$. Then

$$p(1 - p) = 0,$$

so $p = 0$ or $p = 1$.

Final Answer

(a) $\{A_i\}_{i=1}^n$ is a partition iff the events are pairwise disjoint and their union is S .

(b) No.

(c)

$$P(A) = 0 \quad \text{or} \quad P(A) = 1.$$

Problem 4.**15 points**

Suppose that X has probability density function

$$f_X(x) = \begin{cases} cx(2-x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant.

- (a) Find c . (3 points)
- (b) Find the cumulative distribution function $F_X(x)$ for all real x . (5 points)
- (c) Find $P(X > 3/2)$. (2 points)
- (d) Find $\mathbb{E}[X]$ and $\text{Var}(X)$. (5 points)

Solution

(a) A density must integrate to 1, so

$$\begin{aligned} 1 &= c \int_0^2 x(2-x) dx \\ &= c \int_0^2 (2x - x^2) dx \\ &= c \left[x^2 - \frac{x^3}{3} \right]_0^2 = c \left(4 - \frac{8}{3} \right) = c \frac{4}{3}. \end{aligned}$$

Hence

$$c = \frac{3}{4}.$$

(b) For $0 < x < 2$,

$$\begin{aligned} F_X(x) &= \int_0^x \frac{3}{4} t(2-t) dt \\ &= \frac{3}{4} \left[t^2 - \frac{t^3}{3} \right]_0^x \\ &= \frac{3x^2}{4} - \frac{x^3}{4}. \end{aligned}$$

Therefore,

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ \frac{3x^2 - x^3}{4}, & 0 < x < 2, \\ 1, & x \geq 2. \end{cases}$$

(c) Using the CDF,

$$\begin{aligned} P\left(X > \frac{3}{2}\right) &= 1 - F_X\left(\frac{3}{2}\right) \\ &= 1 - \frac{3(3/2)^2 - (3/2)^3}{4} \\ &= 1 - \frac{27}{32} = \frac{5}{32}. \end{aligned}$$

(d) First,

$$\begin{aligned} \mathbb{E}[X] &= \int_0^2 x \frac{3}{4} x(2-x) dx \\ &= \frac{3}{4} \int_0^2 (2x^2 - x^3) dx \\ &= \frac{3}{4} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = 1. \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^2 x^2 \frac{3}{4} x(2-x) dx \\ &= \frac{3}{4} \int_0^2 (2x^3 - x^4) dx \\ &= \frac{3}{4} \left[\frac{x^4}{2} - \frac{x^5}{5} \right]_0^2 \\ &= \frac{6}{5}. \end{aligned}$$

Thus

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{6}{5} - 1 = \frac{1}{5}.$$

Final Answer

$$c = \frac{3}{4}, \quad P(X > 3/2) = \frac{5}{32}, \quad \mathbb{E}[X] = 1, \quad \text{Var}(X) = \frac{1}{5}.$$

The CDF is

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ (3x^2 - x^3)/4, & 0 < x < 2, \\ 1, & x \geq 2. \end{cases}$$

Problem 5.

15 points

A company has found that 40% of customers who visit its website make a purchase. Suppose that 200 customers visit the website independently on a particular day.

Let X denote the number of customers who make a purchase.

1. Determine whether a **Poisson approximation** or a **normal approximation** is more appropriate by verifying the required conditions. Then clearly state the approximating distribution; that is, provide λ if a Poisson approximation is appropriate, or provide μ and σ^2 if a normal approximation is appropriate.
2. Use the approximation from part (a) to estimate each of the following:
 - (a) $\mathbb{P}(70 < X < 90)$
 - (b) $\mathbb{P}(X > 90)$
 - (c) $\mathbb{P}(X = 80)$

Solution

Since each customer independently makes a purchase with probability $p = 0.40$, we have

$$X \sim \text{Bin}(200, 0.40).$$

(1) For a Poisson approximation to be appropriate, n should be large and p should be small. Although $n = 200$ is large, $p = 0.40$ is not small. Therefore, a Poisson approximation is not appropriate. For a normal approximation, we verify that

$$np = 200(0.40) = 80$$

and

$$n(1 - p) = 200(0.60) = 120.$$

Since both quantities are greater than 10, the normal approximation is appropriate. The mean and variance of X are

$$\mu = np = 80$$

and

$$\sigma^2 = np(1 - p) = 200(0.40)(0.60) = 48.$$

Therefore, we approximate X by

$$Y \sim N(80, 48),$$

where

$$\sigma = \sqrt{48} \approx 6.93.$$

(2)

(a) Using the continuity correction,

$$\mathbb{P}(70 < X < 90) \approx \mathbb{P}(70.5 < Y < 89.5).$$

Standardizing,

$$\begin{aligned} \mathbb{P}(70.5 < Y < 89.5) &= \mathbb{P}\left(\frac{70.5 - 80}{\sqrt{48}} < Z < \frac{89.5 - 80}{\sqrt{48}}\right) \\ &\approx \mathbb{P}(-1.37 < Z < 1.37). \end{aligned}$$

Using the standard normal table,

$$\Phi(1.37) = 0.9147$$

and

$$\Phi(-1.37) = 1 - \Phi(1.37) = 0.0853.$$

Therefore,

$$\mathbb{P}(70 < X < 90) \approx 0.9147 - 0.0853 = 0.8294.$$

(b) Using the continuity correction,

$$\mathbb{P}(X > 90) \approx \mathbb{P}(Y > 90.5).$$

Standardizing,

$$\begin{aligned}\mathbb{P}(Y > 90.5) &= \mathbb{P}\left(Z > \frac{90.5 - 80}{\sqrt{48}}\right) \\ &\approx \mathbb{P}(Z > 1.52).\end{aligned}$$

Using the standard normal table,

$$\Phi(1.52) = 0.9357.$$

Thus,

$$\mathbb{P}(X > 90) \approx 1 - \Phi(1.52) = 1 - 0.9357 = 0.0643.$$

(c) Using the continuity correction,

$$\mathbb{P}(X = 80) \approx \mathbb{P}(79.5 < Y < 80.5).$$

Standardizing,

$$\begin{aligned}\mathbb{P}(79.5 < Y < 80.5) &= \mathbb{P}\left(\frac{79.5 - 80}{\sqrt{48}} < Z < \frac{80.5 - 80}{\sqrt{48}}\right) \\ &\approx \mathbb{P}(-0.07 < Z < 0.07).\end{aligned}$$

Using the standard normal table,

$$\Phi(0.07) = 0.5279$$

and

$$\Phi(-0.07) = 0.4721.$$

Therefore,

$$\mathbb{P}(X = 80) \approx 0.5279 - 0.4721 = 0.0558.$$