University of Washington
Department of Mathematics

Midterm Examination Solutions

MATH 394: Probability I

Summer 2026 — Instructor: Arman Jahangiri

Exam date: July 22, 2026
Answering time: 60 minutes
Total: 60 points

Problem 1. **5 points**

A student organization has 7 mathematics graduate students, 5 statistics graduate students, and 6 undergraduate students. A committee of 5 students is selected uniformly at random from all committees of 5.

Find the probability that the committee contains exactly 2 mathematics graduate students, exactly 1 statistics graduate student, and exactly 2 undergraduate students.

Give an exact answer. A correct expression involving binomial coefficients is acceptable.

Solution

There are

$$\binom{18}{5}$$

possible committees of 5 students, and all are equally likely.
To form a committee of the required type, choose:

- 2 of the 7 mathematics graduate students,
- 1 of the 5 statistics graduate students, and
- 2 of the 6 undergraduate students.

Thus the number of favorable committees is

$$\binom{7}{2}\binom{5}{1}\binom{6}{2}.$$

Therefore,

$$P(\text{required composition}) = \frac{\binom{7}{2}\binom{5}{1}\binom{6}{2}}{\binom{18}{5}}.$$

Numerically,

$$\binom{7}{2}\binom{5}{1}\binom{6}{2} = 21 \cdot 5 \cdot 15 = 1575, \qquad \binom{18}{5} = 8568,$$

so

$$\frac{1575}{8568} = \frac{525}{2856} = \frac{175}{952}.$$

Problem 2. **10 points**

A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase "free money" is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention "free money." What is the probability that it is spam?

Solution

(a) Let S be the event that the email is spam, and let F be the event that the email contains the phrase "free money." We are given

$$P(S) = 0.80, \qquad P(F \mid S) = 0.10, \qquad P(F \mid S^c) = 0.01.$$

Bayes' rule gives

$$P(S \mid F) = \frac{P(F \mid S)P(S)}{P(F \mid S)P(S) + P(F \mid S^c)P(S^c)}.$$

Substituting the given values,

$$P(S \mid F) = \frac{0.10 \cdot 0.80}{0.10 \cdot 0.80 + 0.01 \cdot 0.20} = \frac{0.08}{0.082} = \frac{40}{41}.$$

Problem 3. **10 points**

Let A and B be events.

(a) Let S be a sample space and $A_1, \dots, A_n \in \mathcal{F}$ be events. Explain what it means for $A_i's$ to for a partition of S, **(3 points)**

(b) Suppose $P(A) = 0.6$ and $P(B) = 0.5$. Can A and B be disjoint? **(3 points)**

(c) Suppose that A is independent of itself. Use the definition of independence to conclude that $P(A) = 0$ or $P(A) = 1$. **(4 points)**

Solution

(a) The events $A_1, \dots, A_n$ form a partition of the sample space S if they are pairwise disjoint,

$$A_i \cap A_j = \emptyset, \qquad i \neq j,$$

and

$$\bigcup_{i=1}^{n} A_i = S.$$

(b) If A and B were disjoint, then

$$1 \geq P(A \cup B) = P(A) + P(B) = 0.6 + 0.5 = 1.1,$$

which is impossible since probabilities cannot exceed 1. Hence A and B cannot be disjoint.

(c) If A is independent of itself, then

$$P(A \cap A) = P(A)P(A).$$

Since $A \cap A = A$,

$$P(A) = P(A)^2.$$

Let $p = P(A)$. Then

$$p(1-p) = 0,$$

so $p = 0$ or $p = 1$.

Final Answer

(a) $\{A_i\}_{i=1}^n$ is a partition iff the events are pairwise disjoint and their union is S.
(b) No.
(c)

$$P(A) = 0 \quad \text{or} \quad P(A) = 1.$$

Problem 4. **15 points**

Suppose that X has probability density function

$$f_X(x) = \begin{cases} c\,x(2-x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant.

(a) Find c. **(3 points)**

(b) Find the cumulative distribution function $F_X(x)$ for all real x. **(5 points)**

(c) Find $P(X > 3/2)$. **(2 points)**

(d) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. **(5 points)**

Solution

(a) A density must integrate to 1, so

$$\begin{aligned} 1 &= c\int_0^2 x(2-x)\,dx \\ &= c\int_0^2 (2x - x^2)\,dx \\ &= c\left[x^2 - \frac{x^3}{3}\right]_0^2 = c\left(4 - \frac{8}{3}\right) = c\frac{4}{3}. \end{aligned}$$

Hence

$$c = \frac{3}{4}.$$

(b) For $0 < x < 2$,

$$\begin{aligned} F_X(x) &= \int_0^x \frac{3}{4}t(2-t)\,dt \\ &= \frac{3}{4}\left[t^2 - \frac{t^3}{3}\right]_0^x \\ &= \frac{3x^2}{4} - \frac{x^3}{4}. \end{aligned}$$

Therefore,

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{3x^2 - x^3}{4}, & 0 < x < 2, \\ 1, & x \ge 2. \end{cases}$$

(c) Using the CDF,

$$\begin{aligned}P\left(X > \frac{3}{2}\right) &= 1 - F_X\left(\frac{3}{2}\right)\\ &= 1 - \frac{3(3/2)^2 - (3/2)^3}{4}\\ &= 1 - \frac{27}{32} = \frac{5}{32}.\end{aligned}$$

(d) First,

$$\begin{aligned}\mathbb{E}[X] &= \int_0^2 x\frac{3}{4}x(2-x)\,dx\\ &= \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx\\ &= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = 1.\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^2 x^2\frac{3}{4}x(2-x)\,dx\\ &= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx\\ &= \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2\\ &= \frac{6}{5}.\end{aligned}$$

Thus

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \frac{6}{5} - 1 = \frac{1}{5}.$$

Final Answer

$$c = \frac{3}{4}, \qquad P(X > 3/2) = \frac{5}{32}, \qquad \mathbb{E}[X] = 1, \qquad \mathrm{Var}(X) = \frac{1}{5}.$$

The CDF is

$$F_X(x) = \begin{cases} 0, & x \le 0,\\ (3x^2 - x^3)/4, & 0 < x < 2,\\ 1, & x \ge 2.\end{cases}$$

Problem 5. **15 points**

A company has found that 40% of customers who visit its website make a purchase. Suppose that 200 customers visit the website independently on a particular day.

Let X denote the number of customers who make a purchase.

1. Determine whether a **Poisson approximation** or a **normal approximation** is more appropriate by verifying the required conditions. Then clearly state the approximating distribution; that is, provide λ if a Poisson approximation is appropriate, or provide μ and σ^2 if a normal approximation is appropriate. **(5 points)**

2. Use the approximation from part (1) to estimate each of the following: **(10 points)**

   (a)
   $$\mathbb{P}(70 < X < 90)$$
   (3 points)

   (b)
   $$\mathbb{P}(X > 90)$$
   (3 points)

   (c)
   $$\mathbb{P}(X = 80)$$
   (4 points)

Solution

Since each customer independently makes a purchase with probability $p = 0.40$, we have

$$X \sim \text{Bin}(200, 0.40).$$

(1) For a Poisson approximation to be appropriate, n should be large and p should be small. Although

$$n = 200$$

is large, $p = 0.40$ is not small. Therefore, a Poisson approximation is not appropriate.
For a normal approximation, we verify that

$$np = 200(0.40) = 80$$

and

$$n(1-p) = 200(0.60) = 120.$$

Since both quantities are greater than 10, the normal approximation is appropriate.
The mean and variance of X are

$$\mu = np = 80$$

and

$$\sigma^2 = np(1-p) = 200(0.40)(0.60) = 48.$$

Therefore, we approximate X by

$$Y \sim N(80, 48),$$

where
$$\sigma = \sqrt{48} \approx 6.93.$$

(2)
(a) Using the continuity correction,

$$\mathbb{P}(70 < X < 90) \approx \mathbb{P}(70.5 < Y < 89.5).$$

Standardizing,
$$\begin{aligned}\mathbb{P}(70.5 < Y < 89.5) &= \mathbb{P}\left(\frac{70.5-80}{\sqrt{48}} < Z < \frac{89.5-80}{\sqrt{48}}\right)\\ &\approx \mathbb{P}(-1.37 < Z < 1.37).\end{aligned}$$

Using the standard normal table,
$$\Phi(1.37) = 0.9147$$

and
$$\Phi(-1.37) = 1 - \Phi(1.37) = 0.0853.$$

Therefore,
$$\mathbb{P}(70 < X < 90) \approx 0.9147 - 0.0853 = 0.8294.$$

(b) Using the continuity correction,

$$\mathbb{P}(X > 90) \approx \mathbb{P}(Y > 90.5).$$

Standardizing,
$$\begin{aligned}\mathbb{P}(Y > 90.5) &= \mathbb{P}\left(Z > \frac{90.5-80}{\sqrt{48}}\right)\\ &\approx \mathbb{P}(Z > 1.52).\end{aligned}$$

Using the standard normal table,
$$\Phi(1.52) = 0.9357.$$

Thus,
$$\mathbb{P}(X > 90) \approx 1 - \Phi(1.52) = 1 - 0.9357 = 0.0643.$$

(c) Using the continuity correction,

$$\mathbb{P}(X = 80) \approx \mathbb{P}(79.5 < Y < 80.5).$$

Standardizing,
$$\begin{aligned}\mathbb{P}(79.5 < Y < 80.5) &= \mathbb{P}\left(\frac{79.5-80}{\sqrt{48}} < Z < \frac{80.5-80}{\sqrt{48}}\right)\\ &\approx \mathbb{P}(-0.07 < Z < 0.07).\end{aligned}$$

Using the standard normal table,
$$\Phi(0.07) = 0.5279$$

and
$$\Phi(-0.07) = 0.4721.$$

Therefore,
$$\mathbb{P}(X = 80) \approx 0.5279 - 0.4721 = 0.0558.$$