University of Washington
Department of Mathematics

Homework 4

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus**.

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 1.

People are arriving at a party one at a time. While waiting for more people to arrive, they entertain themselves by comparing their birthdays.

Let X be the number of people needed to obtain a birthday match; that is, before person X arrives, no two people have the same birthday, but when person X arrives there is a match.

Find the PMF of X.

Final Answer

Problem 2.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 3.

Let X be a random variable with CDF F, and let

$$Y = \mu + \sigma X,$$

where μ and σ are real numbers with $\sigma > 0$.

The random variable Y is called a *location–scale transformation* of X.

Find the CDF of Y in terms of F.

Final Answer

Problem 3.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 4.

Let n be a positive integer and define

$$F(x) = \frac{\lfloor x \rfloor}{n}$$

for $0 \leq x \leq n$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Also define

$$F(x) = 0 \quad \text{for } x < 0,$$

and

$$F(x) = 1 \quad \text{for } x > n.$$

(a) Show that F is a valid CDF.

(b) Find the PMF corresponding to F.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 4.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 5.

(a) Show that

$$p(n) = \left(\frac{1}{2}\right)^{n+1}, \qquad n = 0, 1, 2, \ldots,$$

is a valid PMF for a discrete random variable.

(b) Find the CDF of a random variable with the PMF from part (a).

Final Answer for Part (a)

Final Answer for Part (b)

Problem 5.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 8.

There are 100 prizes, with one worth \$1, one worth \$2, ..., and one worth \$100. There are 100 boxes, each containing one of the prizes.

You get 5 prizes by picking random boxes one at a time, without replacement.

Find the PMF of how much your most valuable prize is worth, as a simple expression in terms of binomial coefficients.

Final Answer

Problem 6.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 9.

Let F_1 and F_2 be CDFs, let $0 < p < 1$, and define

$$F(x) = pF_1(x) + (1-p)F_2(x)$$

for all x.

(a) Show directly that F has the properties of a valid CDF. The distribution defined by F is called a *mixture* of the distributions defined by F_1 and F_2.

(b) Consider creating a random variable in the following way. Flip a coin with probability p of Heads. If the coin lands Heads, generate a random variable according to F_1; if the coin lands Tails, generate a random variable according to F_2. Show that the random variable obtained in this way has CDF F.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 7.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 22.

There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen, with equal probabilities for the two coins. It is then flipped $n \geq 2$ times.

Let X be the number of times it lands Heads.

(a) Find the PMF of X.

(b) What is the distribution of X if $p_1 = p_2$?

(c) Give an intuitive explanation of why X is not Binomial for $p_1 \neq p_2$. You may assume that n is large for your explanation.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Problem 8.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 30.

A certain company has $n + m$ employees, consisting of n women and m men. The company is deciding which employees to promote.

(a) Suppose for this part that the company decides to promote t employees, where

$$1 \leq t \leq n + m,$$

by choosing t random employees, with equal probabilities for each set of t employees.

What is the distribution of the number of women who get promoted?

(b) Now suppose that instead of having a predetermined number of promotions to give, the company decides independently for each employee, promoting the employee with probability p.

Find the distributions of:

- the number of women who are promoted,
- the number of women who are not promoted,
- the number of employees who are promoted.

(c) In the set-up from part (b), find the conditional distribution of the number of women who are promoted, given that exactly t employees are promoted.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)