University of Washington
Department of Mathematics

Homework 6

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus.**

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1. Conditional Distribution Above a Threshold

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 4.

Let X be a continuous random variable with CDF F and PDF f.

(a) Find the conditional CDF of X given $X > c$, for a constant c with $P(X > c) \neq 0$. That is, find $P(X \leq x \mid X > c)$ for all x, in terms of F.

(b) Find the conditional PDF of X given $X > c$.

(c) Check that the conditional PDF from part (b) is a valid PDF, by showing directly that it is nonnegative and integrates to 1.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Problem 2. Random Radius Circle

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 5.

A circle with a random radius $R \sim \text{Unif}(0, 1)$ is generated. Let A be its area.

(a) Find the mean and variance of A, without first finding the CDF or PDF of A.

(b) Find the CDF and PDF of A.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 3. Longer Piece of a Broken Stick

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 12.

A stick is broken into two pieces, at a uniformly random breakpoint. Find the CDF and average of the length of the longer piece.

Final Answer

Problem 4. Maximum of Uniforms

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 14.

Let $U_1, \ldots, U_n$ be i.i.d. $\text{Unif}(0, 1)$, and

$$X = \max(U_1, \ldots, U_n).$$

What is the PDF of X? What is $E(X)$?

Hint: Find the CDF of X first, by translating the event $X \leq x$ into an event involving $U_1, \ldots, U_n$.

Final Answer

Problem 5. Constructing an Exponential Random Variable

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 15.

Let $U \sim \text{Unif}(0,1)$. Using U, construct

$$X \sim \text{Expo}(\lambda).$$

Final Answer

Problem 6. Measurement Error

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 22.

The distance between two points needs to be measured, in meters. The true distance between the points is 10 meters, but due to measurement error we cannot measure the distance exactly. Instead, we will observe a value of $10 + \epsilon$, where the error ϵ is distributed

$$N(0, 0.04).$$

Find the probability that the observed distance is within 0.4 meters of the true distance, 10 meters. Give both an exact answer in terms of Φ and an approximate numerical answer.

Final Answer

Problem 7. Absolute Value of a Normal

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 29.

Let

$$Y = |X|, \qquad X \sim N(\mu, \sigma^2).$$

This is a well-defined continuous random variable, even though the absolute value function is not differentiable at 0.

(a) Find the CDF of Y in terms of Φ. Be sure to specify the CDF everywhere.

(b) Find the PDF of Y.

(c) Is the PDF of Y continuous at 0? If not, is this a problem as far as using the PDF to find probabilities?

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Problem 8. Post Office with Two Clerks

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 36.

A post office has 2 clerks. Alice enters the post office while 2 other customers, Bob and Claire, are being served by the 2 clerks. She is next in line. Assume that the time a clerk spends serving a customer has an Expo(λ) distribution.

(a) What is the probability that Alice is the last of the 3 customers to be done being served?

Hint: No integrals are needed.

(b) What is the expected total time that Alice needs to spend at the post office?

Final Answer for Part (a)

Final Answer for Part (b)

Problem 9. Exponential as a Limit of Geometric

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 43.

The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times

$$0, \Delta t, 2\Delta t, \ldots,$$

where Δt is a small positive number. Let the probability of success of each trial be $\lambda \Delta t$, where λ is a positive constant. Let G be the number of failures before the first success in discrete time, and let T be the time of the first success in continuous time.

(a) Find a simple equation relating G to T.

Hint: Draw a timeline and try out a simple example.

(b) Find the CDF of T.

Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

Hint: Use the compound interest limit.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Problem 10. Median and Mode

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 1–2.

(a) Let $U \sim \text{Unif}(a, b)$. Find the median and mode of U.

(b) Let $X \sim \text{Expo}(\lambda)$. Find the median and mode of X.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 11. MGF of a Sum of Dice

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 13.

A fair die is rolled twice, with outcomes X for the first roll and Y for the second roll. Find the moment generating function

$$M_{X+Y}(t)$$

of $X + Y$. Your answer should be a function of t and may contain unsimplified finite sums.

Final Answer

Problem 12. Sum of Squared Normals

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 15.

Let

$$W = X^2 + Y^2,$$

with X, Y i.i.d. $N(0,1)$. The MGF of X^2 turns out to be

$$(1-2t)^{-1/2}, \qquad t < \frac{1}{2},$$

and you may assume this.

(a) Find the MGF of W.

(b) What famous distribution that we have studied so far does W follow? Be sure to state the parameters in addition to the name.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 13. Standardized Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 17.

Let $X_1, \dots, X_n$ be i.i.d. with mean μ, variance σ^2, and MGF M. Let

$$Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right).$$

(a) Show that Z_n is a standardized quantity; that is, it has mean 0 and variance 1.

(b) Find the MGF of Z_n in terms of M, the MGF of each X_j.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 14. Poisson Approximation via MGFs

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 21.

Let

$$X_n \sim \text{Bin}(n, p_n)$$

for all $n \geq 1$, where np_n is a constant $\lambda > 0$ for all n, so that

$$p_n = \frac{\lambda}{n}.$$

Let

$$X \sim \text{Pois}(\lambda).$$

Show that the MGF of X_n converges to the MGF of X. This gives another way to see that the $\text{Bin}(n, p)$ distribution can be well-approximated by the $\text{Pois}(\lambda)$ distribution when n is large, p is small, and $\lambda = np$ is moderate.

Final Answer

Problem 15. Triangular Distribution via MGFs

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 23.

Let U_1, U_2 be i.i.d. $\text{Unif}(0,1)$. The random variable $U_1 + U_2$ has a Triangular distribution, with PDF

$$f(t) = \begin{cases} t, & 0 < t \leq 1, \\ 2 - t, & 1 < t < 2. \end{cases}$$

Show that $U_1 + U_2$ has a Triangular distribution by showing that they have the same MGF.

Final Answer