University of Washington
Department of Mathematics

Homework 7

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus**.

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1. Meeting for Lunch

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 1.

Alice and Bob arrange to meet for lunch on a certain day at noon. However, neither is known for punctuality. They both arrive independently at uniformly distributed times between noon and 1 pm on that day. Each is willing to wait up to 15 minutes for the other to show up.

What is the probability they will meet for lunch that day?

Final Answer

Problem 2. Two Breakpoints on a Stick

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 7.

A stick of length L, where L is a positive constant, is broken at a uniformly random point X. Given that $X = x$, another breakpoint Y is chosen uniformly on the interval $[0, x]$.

(a) Find the joint PDF of X and Y. Be sure to specify the support.

(b) We already know that the marginal distribution of X is $\text{Unif}(0, L)$. Check that marginalizing out Y from the joint PDF agrees that this is the marginal distribution of X.

(c) We already know that the conditional distribution of Y given $X = x$ is $\text{Unif}(0, x)$. Check that using the definition of conditional PDFs agrees that this is the conditional distribution of Y given $X = x$.

(d) Find the marginal PDF of Y.

(e) Find the conditional PDF of X given $Y = y$.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Final Answer for Part (d)

Final Answer for Part (e)

Problem 3. Broken Stick and Table Legs

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 14.

(a) A stick is broken into three pieces by picking two points independently and uniformly along the stick, and breaking the stick at those two points. What is the probability that the three pieces can be assembled into a triangle?

Hint: A triangle can be formed from 3 line segments of lengths a, b, c if and only if $a, b, c \in (0, 1/2)$. The probability can be interpreted geometrically as proportional to an area in the plane, avoiding all calculus, but make sure for that approach that the distribution of the random point in the plane is Uniform over some region.

(b) Three legs are positioned uniformly and independently on the perimeter of a round table. What is the probability that the table will stand?

Final Answer for Part (a)

Final Answer for Part (b)

Problem 4. Rectangle Probability from a Joint CDF

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 15.

Let X and Y be continuous random variables, with joint CDF $F(x, y)$. Show that the probability that (X, Y) falls into the rectangle $[a_1, a_2] \times [b_1, b_2]$ is

$$F(a_2, b_2) - F(a_1, b_2) + F(a_1, b_1) - F(a_2, b_1).$$

Final Answer

Problem 5. Joint PDF I

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 16.

Let X and Y have joint PDF

$$f_{X,Y}(x,y) = x + y, \qquad 0 < x < 1,\ 0 < y < 1.$$

(a) Check that this is a valid joint PDF.

(b) Are X and Y independent?

(c) Find the marginal PDFs of X and Y.

(d) Find the conditional PDF of Y given $X = x$.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Final Answer for Part (d)

Problem 6. Joint PDF II

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 17.

Let X and Y have joint PDF

$$f_{X,Y}(x, y) = cxy, \qquad 0 < x < y < 1.$$

(a) Find c to make this a valid joint PDF.

(b) Are X and Y independent?

(c) Find the marginal PDFs of X and Y.

(d) Find the conditional PDF of Y given $X = x$.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Final Answer for Part (d)

Problem 7. Random Quadratic

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 21.

Find the probability that the quadratic polynomial

$$Ax^2 + Bx + 1,$$

where the coefficients A and B are determined by drawing i.i.d. $\text{Unif}(0, 1)$ random variables, has at least one real root.

Hint: By the quadratic formula, the polynomial $ax^2 + bx + c$ *has a real root if and only if* $b^2 - 4ac \geq 0$.

Final Answer

Problem 8. Distance Between Two Uniforms

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 31.

Let X and Y be i.i.d. $\text{Unif}(0, 1)$. Find the standard deviation of the distance between X and Y.

Final Answer

Problem 9. Minimum and Maximum of Geometrics

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 29.

Let X and Y be i.i.d. Geom(p), and let

$$L = \min(X, Y), \qquad M = \max(X, Y).$$

(a) Find the joint PMF of L and M. Are they independent?

(b) Find the marginal distribution of L in two ways: using the joint PMF, and using a story.

(c) Find $E(M)$.

Hint: A quick way is to use part (b) and the fact that $L + M = X + Y$.

(d) Find the joint PMF of L and $M - L$. Are they independent?

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Final Answer for Part (d)

Problem 10. Covariance and Independence

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 39–40.

(a) Two fair, six-sided dice are rolled, one green and one orange, with outcomes X and Y for the green die and the orange die, respectively.

  (i) Compute the covariance of $X + Y$ and $X - Y$.

  (ii) Are $X + Y$ and $X - Y$ independent?

(b) Let X and Y be i.i.d. Unif$(0, 1)$.

  (i) Compute the covariance of $X + Y$ and $X - Y$.

  (ii) Are $X + Y$ and $X - Y$ independent?

Final Answer for Part (a)

Final Answer for Part (b)

Problem 11. Multinomial Covariance

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 7, Problem 65.

Let

$$(X_1, \ldots, X_k)$$

be Multinomial with parameters

$$n \quad \text{and} \quad (p_1, \ldots, p_k).$$

Use indicator random variables to show that

$$\text{Cov}(X_i, X_j) = -np_ip_j, \qquad i \neq j.$$

Final Answer