University of Washington
Department of Mathematics

Homework 8

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus.**

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1. Transformations of Random Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 1–5.

Find the following PDFs.

(a) Find the PDF of e^{-X} for $X \sim \text{Expo}(1)$.

(b) Find the PDF of X^7 for $X \sim \text{Expo}(\lambda)$.

(c) Find the PDF of Z^3 for $Z \sim N(0,1)$.

(d) Find the PDF of Z^4 for $Z \sim N(0,1)$.

(e) Find the PDF of $|Z|$ for $Z \sim N(0,1)$.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Final Answer for Part (d)

Final Answer for Part (e)

Problem 2. Log Ratio of Exponentials

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 13.

Let X and Y be i.i.d. Expo(λ), and let

$$T = \log(X/Y).$$

Find the CDF and PDF of T.

Final Answer

Problem 3. Linear Change of Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 14.

Let X and Y have joint PDF $f_{X,Y}(x,y)$, and transform $(X,Y) \mapsto (T,W)$ linearly by letting

$$T = aX + bY, \qquad W = cX + dY,$$

where a, b, c, d are constants such that

$$ad - bc \neq 0.$$

(a) Find the joint PDF $f_{T,W}(t,w)$ in terms of $f_{X,Y}$, though your answer should be written as a function of t and w.

(b) For the case where

$$T = X + Y, \qquad W = X - Y,$$

show that

$$f_{T,W}(t,w) = \frac{1}{2} f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right).$$

Final Answer for Part (a)

Final Answer for Part (b)

Problem 4. Sum of Independent Gamma Random Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 29.

Let

$$X \sim \text{Gamma}(a, \lambda), \qquad Y \sim \text{Gamma}(b, \lambda),$$

and suppose X and Y are independent, with a and b integers.

Show that

$$X + Y \sim \text{Gamma}(a + b, \lambda)$$

in three ways:

(a) with a convolution integral;

(b) with MGFs;

(c) with a story proof.

Final Answer for Part (a)

Final Answer for Part (b)

Final Answer for Part (c)

Problem 5. A Chebyshev Bound for the Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 2.

For i.i.d. random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2, give a value of n, as a specific number, that will ensure that there is at least a 99% chance that the sample mean will be within 2 standard deviations of the true mean μ.

Final Answer

Problem 6. AM–GM from Jensen's Inequality

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 4.

The famous arithmetic mean–geometric mean inequality says that for any positive numbers

$$a_1, a_2, \ldots, a_n,$$

we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

Show that this inequality follows from Jensen's inequality, by considering $E(\log X)$ for a random variable X whose possible values are $a_1, \ldots, a_n$. You should specify the PMF of X.

Final Answer

Problem 7. Equality and Inequality Symbols

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 7.

Let X and Y be i.i.d. positive random variables, and let $c > 0$. For each part below, fill in the appropriate equality or inequality symbol.

Write $=$ if the two sides are always equal, $\leq$ if the left-hand side is less than or equal to the right-hand side but not necessarily equal, and similarly for $\geq$. If no relation holds in general, write ?.

(a) $E(\log X)$ _____ $\log(EX)$

(b) $E(X)$ _____ $\sqrt{E(X^2)}$

(c) $E(\sin^2 X) + E(\cos^2 X)$ _____ 1

(d) $E(|X|)$ _____ $\sqrt{E(X^2)}$

(e) $P(X > c)$ _____ $\dfrac{E(X^3)}{c^3}$

(f) $P(X \leq Y)$ _____ $P(X \geq Y)$

(g) $E(XY)$ _____ $\sqrt{E(X^2)E(Y^2)}$

(h) $P(X + Y > 10)$ _____ $P(X > 5 \text{ or } Y > 5)$

(i) $E(\min(X, Y))$ _____ $\min(EX, EY)$

(j) $E(X/Y)$ _____ $\dfrac{EX}{EY}$

(k) $E\big(X^2(X^2 + 1)\big)$ _____ $E\big(X^2(Y^2 + 1)\big)$

(l) $E\left(\dfrac{X^3}{X^3 + Y^3}\right)$ _____ $E\left(\dfrac{Y^3}{X^3 + Y^3}\right)$

Final Answer

Problem 8. Ratio of Sample Sums

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 21.

Let $X_1, X_2, \ldots$ be i.i.d. positive random variables with mean 2. Let $Y_1, Y_2, \ldots$ be i.i.d. positive random variables with mean 3.

Show that

$$\frac{X_1 + X_2 + \cdots + X_n}{Y_1 + Y_2 + \cdots + Y_n} \to \frac{2}{3}$$

with probability 1.

Does it matter whether the X_i are independent of the Y_j?

Final Answer

Problem 9. CLT Approximation

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 22.

Let

$$U_1, U_2, \dots, U_{60}$$

be i.i.d. Unif$(0, 1)$ and let

$$X = U_1 + U_2 + \dots + U_{60}.$$

(a) Which important distribution is the distribution of X very close to? Specify what the parameters are, and state which theorem justifies your choice.

(b) Give a simple but accurate approximation for

$$P(X > 17).$$

Justify briefly.

Final Answer for Part (a)

Final Answer for Part (b)

Problem 10. Exponential Transformation and Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 25.

(a) Let
$$Y = e^X, \qquad X \sim \text{Expo}(3).$$
Find the mean and variance of Y.

(b) For $Y_1, \ldots, Y_n$ i.i.d. with the same distribution as Y from part (a), what is the approximate distribution of the sample mean
$$\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$$
when n is large?

Final Answer for Part (a)

Final Answer for Part (b)

Problem 11. MGF of the Poisson Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 27.

Consider i.i.d. Pois(λ) random variables

$$X_1, X_2, \ldots.$$

The MGF of X_j is

$$M(t) = e^{\lambda(e^t-1)}.$$

(a) Find the MGF $M_n(t)$ of the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

(b) Find the limit of $M_n(t)$ as $n \to \infty$. You can do this with almost no calculation using a relevant theorem, or you can use part (a) and the fact that $e^x \approx 1 + x$ if x is very small.

Final Answer for Part (a)

Final Answer for Part (b)