University of Washington
Department of Mathematics

Homework 4 Solutions

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus**.

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 1.

People are arriving at a party one at a time. While waiting for more people to arrive, they entertain themselves by comparing their birthdays.

Let X be the number of people needed to obtain a birthday match; that is, before person X arrives, no two people have the same birthday, but when person X arrives there is a match.

Find the PMF of X.

Solution

Assume the usual birthday model with 365 equally likely birthdays and no leap day. The smallest possible value of X is 2, since one person alone cannot create a match. The largest possible value is 366, since after 365 people with distinct birthdays, the next person must match one of them.

For $X = k$, the first $k - 1$ people must all have distinct birthdays, and the kth person must match one of those $k - 1$ birthdays. The probability that the first $k - 1$ birthdays are distinct is

$$\frac{365 \cdot 364 \cdots (365 - k + 2)}{365^{k-1}}.$$

Once this has happened, the kth person has exactly $k - 1$ favorable birthdays out of 365. Therefore, for $k = 2, 3, \ldots, 366$,

$$P(X = k) = \frac{365 \cdot 364 \cdots (365 - k + 2)}{365^{k-1}} \cdot \frac{k-1}{365}.$$

Final Answer

$$P(X = k) = \frac{365 \cdot 364 \cdots (365 - k + 2)}{365^{k-1}} \cdot \frac{k-1}{365}, \qquad k = 2, 3, \ldots, 366.$$

$$P(X = k) = 0 \qquad \text{otherwise.}$$

Problem 2.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 3.

Let X be a random variable with CDF F, and let

$$Y = \mu + \sigma X,$$

where μ and σ are real numbers with $\sigma > 0$.

The random variable Y is called a *location–scale transformation* of X.

Find the CDF of Y in terms of F.

Solution

Let F_Y denote the CDF of Y. For any real number y,

$$F_Y(y) = P(Y \leq y).$$

Since $Y = \mu + \sigma X$ and $\sigma > 0$, the inequality may be rearranged without reversing its direction:

$$\mu + \sigma X \leq y \qquad \Longleftrightarrow \qquad X \leq \frac{y - \mu}{\sigma}.$$

Thus

$$F_Y(y) = P\left(X \leq \frac{y - \mu}{\sigma}\right) = F\left(\frac{y - \mu}{\sigma}\right).$$

Problem 3.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 4.

Let n be a positive integer and define

$$F(x) = \frac{\lfloor x \rfloor}{n}$$

for $0 \leq x \leq n$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Also define

$$F(x) = 0 \quad \text{for } x < 0,$$

and

$$F(x) = 1 \quad \text{for } x > n.$$

(a) Show that F is a valid CDF.

(b) Find the PMF corresponding to F.

Solution

(a) The function F is nondecreasing because $\lfloor x \rfloor$ is nondecreasing. It also has the correct limiting behavior. For $x < 0$, $F(x) = 0$, so

$$\lim_{x \to -\infty} F(x) = 0.$$

For $x > n$, $F(x) = 1$, so

$$\lim_{x \to \infty} F(x) = 1.$$

The only possible jumps occur at the integers. At an integer j with $1 \leq j \leq n$, the left-hand limit is

$$F(j^-) = \frac{j-1}{n},$$

while

$$F(j) = \frac{j}{n}.$$

Thus the function is right-continuous at each jump. Away from the integers, it is constant on intervals and therefore right-continuous there as well. Hence F has the required properties of a CDF.

(b) The PMF is obtained from the jumps of the CDF. At each integer $j = 1, 2, \ldots, n$,

$$P(X = j) = F(j) - F(j^-) = \frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}.$$

There are no jumps elsewhere, so the probability is zero outside $\{1, 2, \ldots, n\}$.

Problem 4.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 5.

(a) Show that

$$p(n) = \left(\frac{1}{2}\right)^{n+1}, \qquad n = 0, 1, 2, \ldots,$$

is a valid PMF for a discrete random variable.

(b) Find the CDF of a random variable with the PMF from part (a).

Solution

(a) Each value $p(n)$ is nonnegative. It remains to check that the probabilities add to 1. Using the geometric series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

Therefore $p(n)$ is a valid PMF on the nonnegative integers.

(b) Let X have this PMF. If $x < 0$, then no possible value of X is at most x, so $F(x) = 0$. If $x \geq 0$, then

$$F(x) = P(X \leq x) = \sum_{n=0}^{\lfloor x \rfloor} \left(\frac{1}{2}\right)^{n+1}.$$

The finite geometric sum gives

$$F(x) = 1 - \left(\frac{1}{2}\right)^{\lfloor x \rfloor + 1}, \qquad x \geq 0.$$

Final Answer for Part (b)

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \left(\frac{1}{2}\right)^{\lfloor x \rfloor + 1}, & x \geq 0. \end{cases}$$

Problem 5.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 8.

There are 100 prizes, with one worth \$1, one worth \$2, ..., and one worth \$100. There are 100 boxes, each containing one of the prizes.

You get 5 prizes by picking random boxes one at a time, without replacement.

Find the PMF of how much your most valuable prize is worth, as a simple expression in terms of binomial coefficients.

Solution

Let M be the value of the most valuable prize among the 5 prizes selected. The sample space consists of all 5-element subsets of the 100 prizes, so there are

$$\binom{100}{5}$$

equally likely selections.

For the maximum value to be m, the prize worth m dollars must be selected, and the other 4 selected prizes must all come from the prizes worth $1, 2, \ldots, m-1$. This can be done in

$$\binom{m-1}{4}$$

ways. This is possible only when $m \geq 5$.

Therefore

$$P(M = m) = \frac{\binom{m-1}{4}}{\binom{100}{5}}, \qquad m = 5, 6, \ldots, 100.$$

$$P(M = m) = 0 \qquad \text{otherwise.}$$

Problem 6.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 9.

Let F_1 and F_2 be CDFs, let $0 < p < 1$, and define

$$F(x) = pF_1(x) + (1-p)F_2(x)$$

for all x.

(a) Show directly that F has the properties of a valid CDF. The distribution defined by F is called a *mixture* of the distributions defined by F_1 and F_2.

(b) Consider creating a random variable in the following way. Flip a coin with probability p of Heads. If the coin lands Heads, generate a random variable according to F_1; if the coin lands Tails, generate a random variable according to F_2. Show that the random variable obtained in this way has CDF F.

Solution

(a) Since F_1 and F_2 are nondecreasing and p and $1-p$ are positive weights, their weighted average is also nondecreasing. Also,

$$\lim_{x\to-\infty} F(x) = p \lim_{x\to-\infty} F_1(x) + (1-p) \lim_{x\to-\infty} F_2(x) = 0,$$

and

$$\lim_{x\to\infty} F(x) = p \lim_{x\to\infty} F_1(x) + (1-p) \lim_{x\to\infty} F_2(x) = p + (1-p) = 1.$$

Finally, because F_1 and F_2 are right-continuous, for every x,

$$\lim_{t\downarrow x} F(t) = p \lim_{t\downarrow x} F_1(t) + (1-p) \lim_{t\downarrow x} F_2(t) = pF_1(x) + (1-p)F_2(x) = F(x).$$

Thus F is a valid CDF.

Final Answer for Part (a)

$F = pF_1 + (1-p)F_2$ is nondecreasing, right-continuous, tends to 0 at $-\infty$, and tends to 1 at ∞. Hence it is a valid CDF.

(b) Let C be the event that the coin lands Heads. The random variable is generated from F_1 on C and from F_2 on C^c. Therefore, by the law of total probability,

$$P(X \le x) = P(X \le x \mid C)P(C) + P(X \le x \mid C^c)P(C^c).$$

Since $P(C) = p$, $P(C^c) = 1-p$, $P(X \le x \mid C) = F_1(x)$, and $P(X \le x \mid C^c) = F_2(x)$, we get

$$P(X \le x) = pF_1(x) + (1-p)F_2(x) = F(x).$$

So the random variable constructed by this procedure has CDF F.

Final Answer for Part (b)

$$P(X \le x) = pF_1(x) + (1-p)F_2(x) = F(x).$$

Problem 7.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 22.

There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen, with equal probabilities for the two coins. It is then flipped $n \geq 2$ times. Let X be the number of times it lands Heads.

(a) Find the PMF of X.

(b) What is the distribution of X if $p_1 = p_2$?

(c) Give an intuitive explanation of why X is not Binomial for $p_1 \neq p_2$. You may assume that n is large for your explanation.

Solution

(a) Conditional on choosing coin 1, the number of Heads has distribution $\text{Bin}(n, p_1)$. Conditional on choosing coin 2, the number of Heads has distribution $\text{Bin}(n, p_2)$. Since the two coins are chosen with equal probabilities,

$$P(X = k) = \frac{1}{2}\binom{n}{k}p_1^k(1-p_1)^{n-k} + \frac{1}{2}\binom{n}{k}p_2^k(1-p_2)^{n-k},$$

for $k = 0, 1, \ldots, n$.

(b) If $p_1 = p_2 = p$, then the two possible coins have the same Heads probability. The mixture collapses to the ordinary Binomial distribution,

$$P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}.$$

Thus

$$X \sim \text{Bin}(n, p).$$

(c) When $p_1 \neq p_2$, the flips are not independent in the unconditional sense. Seeing many Heads early makes it more likely that the chosen coin is the one with the larger Heads probability, and that changes our prediction for later flips. For large n, the number of Heads tends to cluster around np_1 if coin 1 was chosen and around np_2 if coin 2 was chosen. This is a mixture of two Binomial patterns rather than one Binomial pattern with a single fixed success probability.

Problem 8.

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 3, Problem 30.

A certain company has $n+m$ employees, consisting of n women and m men. The company is deciding which employees to promote.

(a) Suppose for this part that the company decides to promote t employees, where

$$1 \leq t \leq n+m,$$

by choosing t random employees, with equal probabilities for each set of t employees.

What is the distribution of the number of women who get promoted?

(b) Now suppose that instead of having a predetermined number of promotions to give, the company decides independently for each employee, promoting the employee with probability p.

Find the distributions of:

- the number of women who are promoted,
- the number of women who are not promoted,
- the number of employees who are promoted.

(c) In the set-up from part (b), find the conditional distribution of the number of women who are promoted, given that exactly t employees are promoted.

Solution

(a) Let W be the number of women promoted. Since the company chooses t employees from $n+m$ employees without replacement, W has a Hypergeometric distribution. To have $W=k$, the company must choose k of the n women and $t-k$ of the m men. Thus

$$P(W=k) = \frac{\binom{n}{k}\binom{m}{t-k}}{\binom{n+m}{t}},$$

where k must satisfy

$$\max(0, t-m) \leq k \leq \min(n, t).$$

(b) Now each employee is promoted independently with probability p.
The number of women promoted is the number of successes among n independent promotion decisions for women, so

$$W_p \sim \text{Bin}(n, p).$$

The number of women not promoted is the number of failures among those same n women, so

$$W_{np} \sim \text{Bin}(n, 1-p).$$

The total number of employees promoted is the number of successes among all $n+m$ employees, so

$$T \sim \text{Bin}(n+m, p).$$

Final Answer for Part (b)

$$W_p \sim \text{Bin}(n,p), \qquad W_{np} \sim \text{Bin}(n, 1-p), \qquad T \sim \text{Bin}(n+m, p).$$

(c) Given that exactly t employees are promoted, all sets of t promoted employees are equally likely. This brings us back to the same sampling-without-replacement structure as part (a). Therefore the conditional distribution of the number of women promoted is Hypergeometric:

$$P(W = k \mid T = t) = \frac{\binom{n}{k}\binom{m}{t-k}}{\binom{n+m}{t}},$$

for the same possible values of k, i.e., $\max(0, t - m) \le k \le \min(n, t)$.