University of Washington
Department of Mathematics

Homework 6 Solutions

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade.
Further course policies can be seen in the **MATH 394 syllabus.**

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1. Conditional Distribution Above a Threshold

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 4.

Let X be a continuous random variable with CDF F and PDF f.

(a) Find the conditional CDF of X given $X > c$, for a constant c with $P(X > c) \neq 0$. That is, find $P(X \leq x \mid X > c)$ for all x, in terms of F.

(b) Find the conditional PDF of X given $X > c$.

(c) Check that the conditional PDF from part (b) is a valid PDF, by showing directly that it is nonnegative and integrates to 1.

Solution

(a) The event being conditioned on is $\{X > c\}$. If $x \leq c$, then the events $\{X \leq x\}$ and $\{X > c\}$ cannot occur together, so the conditional probability is 0. If $x > c$, then

$$P(X \leq x \mid X > c) = \frac{P(c < X \leq x)}{P(X > c)} = \frac{F(x) - F(c)}{1 - F(c)}.$$

Thus the conditional CDF is

$$F_{X|X>c}(x) = \begin{cases} 0, & x \leq c, \\ \dfrac{F(x) - F(c)}{1 - F(c)}, & x > c. \end{cases}$$

Final Answer for Part (a)

$$F_{X|X>c}(x) = \begin{cases} 0, & x \leq c, \\ \dfrac{F(x) - F(c)}{1 - F(c)}, & x > c. \end{cases}$$

(b) Differentiating the conditional CDF on the region $x > c$ gives the conditional density. Therefore

$$f_{X|X>c}(x) = \begin{cases} \dfrac{f(x)}{1 - F(c)}, & x > c, \\ 0, & x \leq c. \end{cases}$$

Final Answer for Part (b)

$$f_{X|X>c}(x) = \begin{cases} \dfrac{f(x)}{1 - F(c)}, & x > c, \\ 0, & x \leq c. \end{cases}$$

(c) Since $f(x) \geq 0$ and $1 - F(c) = P(X > c) > 0$, the conditional density is nonnegative. Also,

$$\int_{-\infty}^{\infty} f_{X|X>c}(x)\, dx = \int_{c}^{\infty} \frac{f(x)}{1 - F(c)}\, dx = \frac{1 - F(c)}{1 - F(c)} = 1.$$

So the function is indeed a valid PDF.

Final Answer for Part (c)

The conditional density is nonnegative and integrates to 1.

Problem 2. Random Radius Circle

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 5.

A circle with a random radius $R \sim \text{Unif}(0,1)$ is generated. Let A be its area.

(a) Find the mean and variance of A, without first finding the CDF or PDF of A.

(b) Find the CDF and PDF of A.

Solution

The area is

$$A = \pi R^2.$$

(a) Since $R \sim \text{Unif}(0,1)$, we have $E(R^2) = 1/3$ and $E(R^4) = 1/5$. Hence

$$E(A) = \pi E(R^2) = \frac{\pi}{3}.$$

Also,

$$\text{Var}(A) = \pi^2 \text{Var}(R^2) = \pi^2 \left(E(R^4) - [E(R^2)]^2\right) = \pi^2 \left(\frac{1}{5} - \frac{1}{9}\right) = \frac{4\pi^2}{45}.$$

Final Answer for Part (a)

$$E(A) = \frac{\pi}{3}, \qquad \text{Var}(A) = \frac{4\pi^2}{45}.$$

(b) The possible values of A are from 0 to π. For $0 \le a \le \pi$,

$$F_A(a) = P(A \le a) = P(\pi R^2 \le a) = P\left(R \le \sqrt{\frac{a}{\pi}}\right) = \sqrt{\frac{a}{\pi}}.$$

Thus

$$F_A(a) = \begin{cases} 0, & a < 0, \\ \sqrt{a/\pi}, & 0 \le a \le \pi, \\ 1, & a > \pi. \end{cases}$$

Differentiating gives

$$f_A(a) = \begin{cases} \dfrac{1}{2\sqrt{\pi a}}, & 0 < a < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Final Answer for Part (b)

$$F_A(a) = \begin{cases} 0, & a < 0, \\ \sqrt{a/\pi}, & 0 \le a \le \pi, \\ 1, & a > \pi, \end{cases} \qquad f_A(a) = \begin{cases} \dfrac{1}{2\sqrt{\pi a}}, & 0 < a < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3. Longer Piece of a Broken Stick

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 12.

A stick is broken into two pieces, at a uniformly random breakpoint. Find the CDF and average of the length of the longer piece.

Solution

Assume the stick has length 1, and let $U \sim \text{Unif}(0,1)$ be the breakpoint. The two pieces have lengths U and $1-U$, so the longer piece has length

$$L = \max(U, 1-U).$$

The possible values of L lie between $1/2$ and 1.
If $\ell < 1/2$, then $P(L \le \ell) = 0$. If $1/2 \le \ell \le 1$, then

$$L \le \ell \quad \Longleftrightarrow \quad U \le \ell \text{ and } 1-U \le \ell \quad \Longleftrightarrow \quad 1-\ell \le U \le \ell.$$

The length of this interval is $\ell - (1-\ell) = 2\ell - 1$. Thus

$$F_L(\ell) = \begin{cases} 0, & \ell < 1/2, \\ 2\ell - 1, & 1/2 \le \ell \le 1, \\ 1, & \ell > 1. \end{cases}$$

This means that L is uniform on $[1/2, 1]$, so its average is the midpoint of that interval,

$$E(L) = \frac{1/2 + 1}{2} = \frac{3}{4}.$$

Final Answer

$$F_L(\ell) = \begin{cases} 0, & \ell < 1/2, \\ 2\ell - 1, & 1/2 \le \ell \le 1, \\ 1, & \ell > 1, \end{cases} \qquad E(L) = \frac{3}{4}.$$

Problem 4. Maximum of Uniforms

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 14.

Let $U_1, \ldots, U_n$ be i.i.d. $\text{Unif}(0,1)$, and

$$X = \max(U_1, \ldots, U_n).$$

What is the PDF of X? What is $E(X)$?

Hint: Find the CDF of X first, by translating the event $X \le x$ into an event involving $U_1, \ldots, U_n$.

Solution

For $0 \le x \le 1$,

$$F_X(x) = P(X \le x) = P(U_1 \le x, \ldots, U_n \le x) = x^n,$$

since the U_j are independent. Therefore

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^n, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

Differentiating on $(0,1)$ gives

$$f_X(x) = nx^{n-1}, \qquad 0 < x < 1.$$

The expectation is

$$E(X) = \int_0^1 x\, nx^{n-1}\, dx = n \int_0^1 x^n\, dx = \frac{n}{n+1}.$$

Final Answer

$$f_X(x) = \begin{cases} nx^{n-1}, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad E(X) = \frac{n}{n+1}.$$

Problem 5. Constructing an Exponential Random Variable

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 15.

Let $U \sim \text{Unif}(0,1)$. Using U, construct
$$X \sim \text{Expo}(\lambda).$$

Solution

The exponential CDF is
$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0.$$
To use inverse transform sampling, set
$$U = 1 - e^{-\lambda X}.$$
Solving for X gives
$$X = -\frac{1}{\lambda}\log(1-U).$$
Since $1-U$ is also uniform on $(0,1)$, the equivalent construction
$$X = -\frac{1}{\lambda}\log U$$
is also valid.

Final Answer

$$X = -\frac{1}{\lambda}\log(1-U)$$
Equivalently,
$$X = -\frac{1}{\lambda}\log U.$$

Problem 6. Measurement Error

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 22.

The distance between two points needs to be measured, in meters. The true distance between the points is 10 meters, but due to measurement error we cannot measure the distance exactly. Instead, we will observe a value of $10 + \epsilon$, where the error ϵ is distributed

$$N(0, 0.04).$$

Find the probability that the observed distance is within 0.4 meters of the true distance, 10 meters. Give both an exact answer in terms of Φ and an approximate numerical answer.

Solution

Being within 0.4 meters of the true value means

$$|\epsilon| \leq 0.4.$$

Since $\epsilon \sim N(0, 0.04)$, its standard deviation is 0.2. Thus

$$P(|\epsilon| \leq 0.4) = P\left(-2 \leq \frac{\epsilon}{0.2} \leq 2\right) = \Phi(2) - \Phi(-2).$$

By symmetry, this is

$$2\Phi(2) - 1.$$

Using $\Phi(2) \approx 0.97725$, the probability is approximately

$$2(0.97725) - 1 = 0.9545.$$

Final Answer

$$P(|\epsilon| \leq 0.4) = 2\Phi(2) - 1 \approx 0.9545.$$

Problem 7. Absolute Value of a Normal

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 29.

Let
$$Y = |X|, \qquad X \sim N(\mu, \sigma^2).$$
This is a well-defined continuous random variable, even though the absolute value function is not differentiable at 0.

(a) Find the CDF of Y in terms of Φ. Be sure to specify the CDF everywhere.

(b) Find the PDF of Y.

(c) Is the PDF of Y continuous at 0? If not, is this a problem as far as using the PDF to find probabilities?

Solution

(a) Since $Y = |X|$, the random variable Y is always nonnegative. Thus $F_Y(y) = 0$ for $y < 0$. For $y \geq 0$,
$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y).$$
Standardizing gives
$$F_Y(y) = \Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{-y-\mu}{\sigma}\right).$$

Final Answer for Part (a)

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \Phi\left(\dfrac{y-\mu}{\sigma}\right) - \Phi\left(\dfrac{-y-\mu}{\sigma}\right), & y \geq 0. \end{cases}$$

(b) Differentiating for $y > 0$ gives
$$f_Y(y) = \frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right) + \frac{1}{\sigma}\phi\left(\frac{-y-\mu}{\sigma}\right), \qquad y > 0,$$
and $f_Y(y) = 0$ for $y < 0$.

Final Answer for Part (b)

$$f_Y(y) = \begin{cases} \dfrac{1}{\sigma}\phi\left(\dfrac{y-\mu}{\sigma}\right) + \dfrac{1}{\sigma}\phi\left(\dfrac{-y-\mu}{\sigma}\right), & y > 0, \\ 0, & y < 0. \end{cases}$$

(c) At 0, the density has a jump if we define it to be 0 to the left and by the right-hand formula to the right. This is not a problem. A PDF only needs to integrate correctly, and changing the value of a density at a single point does not change any probability.

Final Answer for Part (c)

The PDF need not be continuous at 0, and this causes no issue for computing probabilities.

Problem 8. Post Office with Two Clerks

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 36.

A post office has 2 clerks. Alice enters the post office while 2 other customers, Bob and Claire, are being served by the 2 clerks. She is next in line. Assume that the time a clerk spends serving a customer has an Expo(λ) distribution.

(a) What is the probability that Alice is the last of the 3 customers to be done being served?

Hint: No integrals are needed.

(b) What is the expected total time that Alice needs to spend at the post office?

Solution

(a) By the memoryless property of the exponential distribution, once one of Bob or Claire finishes, Alice begins service while the other remaining original customer still has an exponential remaining service time with rate λ. At that moment Alice and the remaining customer are in a perfectly symmetric race, with independent Expo(λ) service times. Therefore Alice is last exactly if her own service time exceeds the other remaining service time, which has probability $1/2$.

Final Answer for Part (a)

$$P(\text{Alice is last}) = \frac{1}{2}.$$

(b) Alice first waits until one of the two clerks becomes free. The minimum of two independent Expo(λ) random variables is Expo(2λ), so her expected waiting time is

$$\frac{1}{2\lambda}.$$

After that, Alice receives service, whose expected duration is

$$\frac{1}{\lambda}.$$

Hence her expected total time in the post office is

$$\frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}.$$

Final Answer for Part (b)

$$E(\text{Alice's total time}) = \frac{3}{2\lambda}.$$

Problem 9. Exponential as a Limit of Geometric

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 5, Problem 43.

The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times

$$0, \Delta t, 2\Delta t, \ldots,$$

where Δt is a small positive number. Let the probability of success of each trial be $\lambda \Delta t$, where λ is a positive constant. Let G be the number of failures before the first success in discrete time, and let T be the time of the first success in continuous time.

(a) Find a simple equation relating G to T.

Hint: Draw a timeline and try out a simple example.

(b) Find the CDF of T.

Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

Hint: Use the compound interest limit.

Solution

(a) If there are G failures before the first success, then the first success occurs at the next trial time. Since trial times are spaced by Δt, we have

$$T = G\Delta t.$$

This convention regards the trial at time 0 as the first possible trial, so $G = 0$ corresponds to success at time 0.

Final Answer for Part (a)

$$T = G\Delta t.$$

(b) For a fixed $t \geq 0$, the event $T > t$ means that all trials up to time t have failed. The number of such trial opportunities is naturally written using $\lfloor t/\Delta t \rfloor + 1$ under the convention that trials begin at time 0. Thus

$$P(T > t) = (1 - \lambda \Delta t)^{\lfloor t/\Delta t \rfloor + 1}.$$

Therefore

$$F_T(t) = P(T \leq t) = 1 - (1 - \lambda \Delta t)^{\lfloor t/\Delta t \rfloor + 1}, \qquad t \geq 0.$$

For $t < 0$, $F_T(t) = 0$.

Final Answer for Part (b)

$$F_T(t) = \begin{cases} 0, & t < 0, \\ 1 - (1 - \lambda \Delta t)^{\lfloor t/\Delta t \rfloor + 1}, & t \geq 0. \end{cases}$$

(c) For fixed $t \geq 0$, as $\Delta t \to 0$,

$$(1 - \lambda \Delta t)^{\lfloor t/\Delta t \rfloor + 1} \longrightarrow e^{-\lambda t}.$$

Hence

$$F_T(t) \longrightarrow 1 - e^{-\lambda t},$$

which is exactly the CDF of an Expo(λ) random variable.

Final Answer for Part (c)

$$F_T(t) \to 1 - e^{-\lambda t}, \qquad t \geq 0.$$

Problem 10. Median and Mode

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 1–2.

(a) Let $U \sim \text{Unif}(a, b)$. Find the median and mode of U.

(b) Let $X \sim \text{Expo}(\lambda)$. Find the median and mode of X.

Solution

(a) The uniform distribution on $[a, b]$ is symmetric, and its CDF equals $1/2$ at the midpoint. Hence the median is

$$\frac{a+b}{2}.$$

The density is constant on the entire interval $[a, b]$, so every point in $[a, b]$ is a mode.

Final Answer for Part (a)

Median: $\frac{a+b}{2}$. Every point in $[a, b]$ is a mode.

(b) For $X \sim \text{Expo}(\lambda)$, the median m solves

$$1 - e^{-\lambda m} = \frac{1}{2}.$$

Thus

$$e^{-\lambda m} = \frac{1}{2}, \qquad m = \frac{\log 2}{\lambda}.$$

The exponential density $\lambda e^{-\lambda x}$ is decreasing on $[0, \infty)$, so its maximum occurs at $x = 0$.

Final Answer for Part (b)

$$\text{Median} = \frac{\log 2}{\lambda}, \qquad \text{Mode} = 0.$$

Problem 11. MGF of a Sum of Dice

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 13.

A fair die is rolled twice, with outcomes X for the first roll and Y for the second roll. Find the moment generating function

$$M_{X+Y}(t)$$

of $X+Y$. Your answer should be a function of t and may contain unsimplified finite sums.

Solution

Since the two die rolls are independent,

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

The two rolls have the same distribution, so

$$M_X(t) = M_Y(t) = E(e^{tX}) = \frac{1}{6}\sum_{j=1}^{6} e^{tj}.$$

Therefore

$$M_{X+Y}(t) = \left(\frac{1}{6}\sum_{j=1}^{6} e^{tj}\right)^2.$$

Equivalently,

$$M_{X+Y}(t) = \frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6} e^{t(i+j)}.$$

Final Answer

$$M_{X+Y}(t) = \left(\frac{1}{6}\sum_{j=1}^{6} e^{tj}\right)^2.$$

Problem 12. Sum of Squared Normals

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 15.

Let

$$W = X^2 + Y^2,$$

with X, Y i.i.d. $N(0,1)$. The MGF of X^2 turns out to be

$$(1-2t)^{-1/2}, \qquad t < \frac{1}{2},$$

and you may assume this.

(a) Find the MGF of W.

(b) What famous distribution that we have studied so far does W follow? Be sure to state the parameters in addition to the name.

Solution

(a) Since X and Y are independent, X^2 and Y^2 are independent. Therefore

$$M_W(t) = M_{X^2}(t) M_{Y^2}(t) = (1-2t)^{-1/2}(1-2t)^{-1/2} = (1-2t)^{-1},$$

for $t < 1/2$.

Final Answer for Part (a)

$$M_W(t) = (1-2t)^{-1}, \qquad t < \frac{1}{2}.$$

(b) The MGF

$$(1-2t)^{-1}$$

is the MGF of a Gamma$(1, 1/2)$ distribution if we use the shape–rate convention, or equivalently an Expo$(1/2)$ distribution. It is also the chi-square distribution with 2 degrees of freedom.

Final Answer for Part (b)

$$W \sim \chi^2_2,$$

equivalently,

$$W \sim \text{Expo}\left(\frac{1}{2}\right).$$

Problem 13. Standardized Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 17.

Let $X_1, \ldots, X_n$ be i.i.d. with mean μ, variance σ^2, and MGF M. Let

$$Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right).$$

(a) Show that Z_n is a standardized quantity; that is, it has mean 0 and variance 1.

(b) Find the MGF of Z_n in terms of M, the MGF of each X_j.

Solution

(a) Since $E(\bar{X}_n) = \mu$, we have

$$E(Z_n) = \frac{\sqrt{n}}{\sigma}(E(\bar{X}_n) - \mu) = 0.$$

Also,

$$\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

Therefore

$$\mathrm{Var}(Z_n) = \frac{n}{\sigma^2}\,\mathrm{Var}(\bar{X}_n) = \frac{n}{\sigma^2} \cdot \frac{\sigma^2}{n} = 1.$$

Final Answer for Part (a)

$$E(Z_n) = 0, \qquad \mathrm{Var}(Z_n) = 1.$$

(b) Write

$$Z_n = \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu).$$

Then

$$M_{Z_n}(t) = E\left[\exp\left(\frac{t}{\sigma\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu)\right)\right].$$

Using independence, this becomes

$$\prod_{j=1}^{n} E\left[\exp\left(\frac{t}{\sigma\sqrt{n}}(X_j - \mu)\right)\right] = \left(e^{-\mu t/(\sigma\sqrt{n})} M\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n.$$

Final Answer for Part (b)

$$M_{Z_n}(t) = \left(e^{-\mu t/(\sigma\sqrt{n})} M\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n.$$

Problem 14. Poisson Approximation via MGFs

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 21.

Let

$$X_n \sim \text{Bin}(n, p_n)$$

for all $n \geq 1$, where np_n is a constant $\lambda > 0$ for all n, so that

$$p_n = \frac{\lambda}{n}.$$

Let

$$X \sim \text{Pois}(\lambda).$$

Show that the MGF of X_n converges to the MGF of X. This gives another way to see that the $\text{Bin}(n, p)$ distribution can be well-approximated by the $\text{Pois}(\lambda)$ distribution when n is large, p is small, and $\lambda = np$ is moderate.

Solution

The MGF of a $\text{Bin}(n, p_n)$ random variable is

$$M_{X_n}(t) = \left(1 - p_n + p_n e^t\right)^n.$$

Since $p_n = \lambda/n$,

$$M_{X_n}(t) = \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n.$$

As $n \to \infty$, we use the standard limit

$$\left(1 + \frac{a}{n}\right)^n \to e^a.$$

With $a = \lambda(e^t - 1)$, this gives

$$M_{X_n}(t) \to \exp\{\lambda(e^t - 1)\}.$$

But

$$M_X(t) = \exp\{\lambda(e^t - 1)\}$$

is the MGF of $X \sim \text{Pois}(\lambda)$.

Final Answer

$$M_{X_n}(t) = \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n \longrightarrow \exp\{\lambda(e^t - 1)\} = M_X(t).$$

Problem 15. Triangular Distribution via MGFs

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 6, Problem 23.

Let U_1, U_2 be i.i.d. $\text{Unif}(0,1)$. The random variable $U_1 + U_2$ has a Triangular distribution, with PDF

$$f(t) = \begin{cases} t, & 0 < t \leq 1, \\ 2-t, & 1 < t < 2. \end{cases}$$

Show that $U_1 + U_2$ has a Triangular distribution by showing that they have the same MGF.

Solution

For $U \sim \text{Unif}(0,1)$,

$$M_U(s) = E(e^{sU}) = \int_0^1 e^{su}\,du = \frac{e^s - 1}{s},$$

with the value at $s = 0$ understood by continuity. Since U_1 and U_2 are independent,

$$M_{U_1+U_2}(s) = \left(\frac{e^s - 1}{s}\right)^2.$$

Now compute the MGF of the triangular density. It is

$$M(s) = \int_0^1 e^{st} t\,dt + \int_1^2 e^{st}(2-t)\,dt.$$

A direct integration gives

$$\int_0^1 e^{st} t\,dt = \frac{e^s(s-1)+1}{s^2},$$

and

$$\int_1^2 e^{st}(2-t)\,dt = \frac{e^{2s} - e^s(s+1)}{s^2}.$$

Adding these two expressions,

$$M(s) = \frac{e^{2s} - 2e^s + 1}{s^2} = \left(\frac{e^s - 1}{s}\right)^2.$$

This is exactly the MGF of $U_1 + U_2$. Since MGFs determine distributions when they exist in a neighborhood of 0, $U_1 + U_2$ has the stated triangular distribution.

Final Answer

$$M_{U_1+U_2}(s) = \left(\frac{e^s - 1}{s}\right)^2,$$

which equals the MGF obtained from the triangular density.