University of Washington
Department of Mathematics

Homework 8 Solutions

MATH 394: Probability I

Instructor: Arman Jahangiri

Submission: Single PDF on Gradescope
Deadline: 10:00 PM Pacific Time on the listed due date

This homework is worth **50 points**.
Across the quarter, there are **eight homework assignments**, worth **400 points total**. Homework assignments together account for **40%** of the final course grade. Further course policies can be seen in the **MATH 394 syllabus**.

Homework Policy

Submission and Deadline

Please:

- Submit homework as a single PDF on Gradescope by **10:00 PM (Pacific Time)** on the listed due date.
- Begin each problem on a new page, and ensure that pages are correctly tagged in Gradescope.
- Write solutions clearly and legibly.
- Typesetting solutions in LaTeX is encouraged, but not required.

Late Days

Each student is allotted **six late days** for the quarter. A late day extends a homework deadline by up to 24 hours without penalty. For example, submitting an assignment anytime between 10:01 PM on the due date and 10:00 PM the following day counts as one late day.

The following rules apply:

- No explanation is required to use late days.
- At most **two late days** may be used on any single homework assignment.
- Late days may not be used on the final homework assignment or on any assignment for which solutions have already been released.
- It is the student's responsibility to keep track of how many late days have been used during the quarter.

Once all late days have been exhausted, additional late submissions will incur a penalty of **10% per day**, up to a maximum deduction of **50%**. Assignments submitted more than five days late, or after solutions have been released, will not be accepted.

Submission Issues and Technical Difficulties

If a serious technical issue prevents a timely Gradescope submission, students may temporarily submit their assignment by email to the instructor at armanjg@uw.edu. In such cases:

- The submission must consist of a **single PDF file**. The email subject line must follow the format:

  `HW# Submission - FULL NAME - STUDENT ID.`

- Once the Gradescope issue has been resolved, the student must upload the *exact same file* to Gradescope as soon as possible.
- The student should clearly indicate in the Gradescope submission that the assignment was previously submitted by email before the deadline.

Email submissions are intended only for genuine technical emergencies and should not be used as a substitute for timely Gradescope submission.

Problem 1. Transformations of Random Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 1–5.

Find the following PDFs.

(a) Find the PDF of e^{-X} for $X \sim \text{Expo}(1)$.

(b) Find the PDF of X^7 for $X \sim \text{Expo}(\lambda)$.

(c) Find the PDF of Z^3 for $Z \sim N(0,1)$.

(d) Find the PDF of Z^4 for $Z \sim N(0,1)$.

(e) Find the PDF of $|Z|$ for $Z \sim N(0,1)$.

Solution

Let $\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$ denote the standard normal density.

(a) Put $Y = e^{-X}$. Since $X > 0$, we have $0 < Y < 1$. The inverse transformation is $x = -\log y$, and

$$\left|\frac{dx}{dy}\right| = \frac{1}{y}.$$

Thus

$$f_Y(y) = e^{-(-\log y)}\frac{1}{y} = 1, \qquad 0 < y < 1.$$

Final Answer for Part (a)

$$f_Y(y) = 1, \qquad 0 < y < 1.$$

(b) Put $Y = X^7$. The inverse is $x = y^{1/7}$ for $y > 0$, and

$$\left|\frac{dx}{dy}\right| = \frac{1}{7}y^{-6/7}.$$

Therefore

$$f_Y(y) = \frac{\lambda}{7}y^{-6/7}e^{-\lambda y^{1/7}}, \qquad y > 0.$$

Final Answer for Part (b)

$$f_Y(y) = \frac{\lambda}{7}y^{-6/7}e^{-\lambda y^{1/7}}, \qquad y > 0.$$

(c) Put $Y = Z^3$. This transformation is one-to-one. For $y \neq 0$, the inverse is $z = y^{1/3}$ and

$$\left|\frac{dz}{dy}\right| = \frac{1}{3|y|^{2/3}}.$$

Hence

$$f_Y(y) = \frac{\phi(y^{1/3})}{3|y|^{2/3}}, \qquad y \neq 0.$$

The value assigned to the density at the single point $y = 0$ is immaterial.

Final Answer for Part (c)

$$f_Y(y) = \frac{\phi(y^{1/3})}{3|y|^{2/3}}, \qquad y \neq 0.$$

(d) Put $Y = Z^4$. The support is $y > 0$. For each $y > 0$, the two preimages are $y^{1/4}$ and $-y^{1/4}$. Adding the two contributions gives

$$f_Y(y) = \phi(y^{1/4})\frac{1}{4y^{3/4}} + \phi(-y^{1/4})\frac{1}{4y^{3/4}} = \frac{\phi(y^{1/4})}{2y^{3/4}}, \qquad y > 0.$$

Final Answer for Part (d)

$$f_Y(y) = \frac{\phi(y^{1/4})}{2y^{3/4}}, \qquad y > 0.$$

(e) Put $Y = |Z|$. For $y > 0$, the two preimages are y and $-y$, and therefore

$$f_Y(y) = \phi(y) + \phi(-y) = 2\phi(y), \qquad y > 0.$$

Final Answer for Part (e)

$$f_Y(y) = 2\phi(y), \qquad y > 0.$$

Problem 2. Log Ratio of Exponentials

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 13.

Let X and Y be i.i.d. Expo(λ), and let

$$T = \log(X/Y).$$

Find the CDF and PDF of T.

Solution

For real t,

$$F_T(t) = P(\log(X/Y) \le t) = P(X \le e^t Y).$$

Conditioning on $Y = y$ gives

$$F_T(t) = \int_0^\infty P(X \le e^t y)\lambda e^{-\lambda y}\, dy.$$

Since $X \sim \text{Expo}(\lambda)$,

$$P(X \le e^t y) = 1 - e^{-\lambda e^t y}.$$

Thus

$$F_T(t) = \int_0^\infty \left(1 - e^{-\lambda e^t y}\right) \lambda e^{-\lambda y}\, dy = 1 - \frac{1}{1+e^t} = \frac{e^t}{1+e^t}.$$

Differentiating the CDF gives

$$f_T(t) = \frac{e^t}{(1+e^t)^2}, \qquad -\infty < t < \infty.$$

The parameter λ disappears because only the ratio X/Y matters.

Final Answer

$$F_T(t) = \frac{e^t}{1+e^t}, \qquad f_T(t) = \frac{e^t}{(1+e^t)^2}, \qquad -\infty < t < \infty.$$

Problem 3. Linear Change of Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 14.

Let X and Y have joint PDF $f_{X,Y}(x,y)$, and transform $(X,Y) \mapsto (T,W)$ linearly by letting

$$T = aX + bY, \qquad W = cX + dY,$$

where a,b,c,d are constants such that

$$ad - bc \neq 0.$$

(a) Find the joint PDF $f_{T,W}(t,w)$ in terms of $f_{X,Y}$, though your answer should be written as a function of t and w.

(b) For the case where

$$T = X + Y, \qquad W = X - Y,$$

show that

$$f_{T,W}(t,w) = \frac{1}{2} f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right).$$

Solution

(a) Let

$$D = ad - bc.$$

Solving the linear system gives

$$x = \frac{dt - bw}{D}, \qquad y = \frac{-ct + aw}{D}.$$

The Jacobian determinant of the inverse transformation is

$$\left|\frac{\partial(x,y)}{\partial(t,w)}\right| = \frac{1}{|D|}.$$

Therefore

$$f_{T,W}(t,w) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dt-bw}{ad-bc}, \frac{-ct+aw}{ad-bc}\right),$$

for all (t,w) whose inverse point lies in the support of (X,Y).

Final Answer for Part (a)

$$f_{T,W}(t,w) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dt-bw}{ad-bc}, \frac{-ct+aw}{ad-bc}\right).$$

(b) Here $a = 1$, $b = 1$, $c = 1$, and $d = -1$, so

$$ad - bc = -2.$$

Also

$$x = \frac{t+w}{2}, \qquad y = \frac{t-w}{2}.$$

Thus

$$f_{T,W}(t,w) = \frac{1}{2} f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right).$$

Final Answer for Part (b)

$$f_{T,W}(t,w) = \frac{1}{2} f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right).$$

Problem 4. Sum of Independent Gamma Random Variables

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 8, Problem 29.

Let

$$X \sim \text{Gamma}(a, \lambda), \qquad Y \sim \text{Gamma}(b, \lambda),$$

and suppose X and Y are independent, with a and b integers.

Show that

$$X + Y \sim \text{Gamma}(a + b, \lambda)$$

in three ways:

(a) with a convolution integral;

(b) with MGFs;

(c) with a story proof.

Solution

(a) For $s > 0$, the convolution density of $S = X + Y$ is

$$f_S(s) = \int_0^s \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \frac{\lambda^b}{\Gamma(b)} (s - x)^{b-1} e^{-\lambda(s-x)}\, dx.$$

This becomes

$$f_S(s) = \frac{\lambda^{a+b} e^{-\lambda s}}{\Gamma(a)\Gamma(b)} \int_0^s x^{a-1}(s - x)^{b-1}\, dx.$$

Using the beta integral,

$$\int_0^s x^{a-1}(s - x)^{b-1}\, dx = s^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.$$

Thus

$$f_S(s) = \frac{\lambda^{a+b}}{\Gamma(a + b)} s^{a+b-1} e^{-\lambda s}, \qquad s > 0,$$

which is the Gamma$(a + b, \lambda)$ density.

Final Answer for Part (a)

$$X + Y \sim \text{Gamma}(a + b, \lambda).$$

(b) The MGF of Gamma(a, λ), using the rate parameter λ, is

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^a, \qquad t < \lambda.$$

By independence,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^a \left(\frac{\lambda}{\lambda - t}\right)^b = \left(\frac{\lambda}{\lambda - t}\right)^{a+b}.$$

This is the MGF of Gamma$(a + b, \lambda)$.

Final Answer for Part (b)

$$M_{X+Y}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{a+b}.$$

(c) When a and b are positive integers, $\mathrm{Gamma}(a, \lambda)$ is the waiting time for the ath arrival in a Poisson process of rate λ. Likewise, $\mathrm{Gamma}(b, \lambda)$ is the additional waiting time for the next b arrivals. By the independent increments property of the Poisson process, the total time to wait for $a + b$ arrivals is the sum of these two independent waiting times. Hence the total waiting time is $\mathrm{Gamma}(a + b, \lambda)$.

Final Answer for Part (c)

$$X + Y \sim \mathrm{Gamma}(a + b, \lambda).$$

Problem 5. A Chebyshev Bound for the Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 2.

For i.i.d. random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2, give a value of n, as a specific number, that will ensure that there is at least a 99% chance that the sample mean will be within 2 standard deviations of the true mean μ.

Solution

The sample mean has

$$E(\bar{X}_n) = \mu, \qquad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

Chebyshev's inequality gives

$$P(|\bar{X}_n - \mu| \geq 2\sigma) \leq \frac{\mathrm{Var}(\bar{X}_n)}{(2\sigma)^2} = \frac{1}{4n}.$$

To make the probability of being outside the interval at most 0.01, it is enough to require

$$\frac{1}{4n} \leq 0.01.$$

Thus $n \geq 25$ is sufficient. With $n = 25$, Chebyshev gives

$$P(|\bar{X}_n - \mu| < 2\sigma) \geq 0.99.$$

Final Answer

$$n = 25$$

Problem 6. AM–GM from Jensen's Inequality

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 4.

The famous arithmetic mean–geometric mean inequality says that for any positive numbers

$$a_1, a_2, \ldots, a_n,$$

we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

Show that this inequality follows from Jensen's inequality, by considering $E(\log X)$ for a random variable X whose possible values are $a_1, \ldots, a_n$. You should specify the PMF of X.

Solution

Let X take the values $a_1, a_2, \ldots, a_n$ with equal probabilities:

$$P(X = a_j) = \frac{1}{n}, \qquad j = 1, \ldots, n.$$

Then

$$E(X) = \frac{a_1 + \cdots + a_n}{n}$$

and

$$E(\log X) = \frac{1}{n} \sum_{j=1}^{n} \log a_j = \log \left((a_1 a_2 \cdots a_n)^{1/n} \right).$$

Since $\log x$ is concave, Jensen's inequality gives

$$E(\log X) \leq \log(EX).$$

Therefore

$$\log \left((a_1 a_2 \cdots a_n)^{1/n} \right) \leq \log \left(\frac{a_1 + \cdots + a_n}{n} \right).$$

Because the logarithm is increasing, exponentiating both sides gives

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Final Answer

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

Problem 7. Equality and Inequality Symbols

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 7.

Let X and Y be i.i.d. positive random variables, and let $c > 0$. For each part below, fill in the appropriate equality or inequality symbol.

Write $=$ if the two sides are always equal, $\leq$ if the left-hand side is less than or equal to the right-hand side but not necessarily equal, and similarly for $\geq$. If no relation holds in general, write ?.

(a) $E(\log X)$ _____ $\log(EX)$

(b) $E(X)$ _____ $\sqrt{E(X^2)}$

(c) $E(\sin^2 X) + E(\cos^2 X)$ _____ 1

(d) $E(|X|)$ _____ $\sqrt{E(X^2)}$

(e) $P(X > c)$ _____ $\dfrac{E(X^3)}{c^3}$

(f) $P(X \leq Y)$ _____ $P(X \geq Y)$

(g) $E(XY)$ _____ $\sqrt{E(X^2)E(Y^2)}$

(h) $P(X + Y > 10)$ _____ $P(X > 5 \text{ or } Y > 5)$

(i) $E(\min(X, Y))$ _____ $\min(EX, EY)$

(j) $E(X/Y)$ _____ $\dfrac{EX}{EY}$

(k) $E\big(X^2(X^2 + 1)\big)$ _____ $E\big(X^2(Y^2 + 1)\big)$

(l) $E\left(\dfrac{X^3}{X^3 + Y^3}\right)$ _____ $E\left(\dfrac{Y^3}{X^3 + Y^3}\right)$

Solution

The answers are as follows.

Part	Symbol	Reason
(a)	$\leq$	$\log x$ is concave, by Jensen.
(b)	$\leq$	$(EX)^2 \leq E(X^2)$.
(c)	$=$	$\sin^2 X + \cos^2 X = 1$ pointwise.
(d)	$\leq$	$(E\|X\|)^2 \leq E(X^2)$.
(e)	$\leq$	Markov applied to X^3.
(f)	$=$	Symmetry of i.i.d. X, Y.
(g)	$\leq$	Cauchy–Schwarz.
(h)	$\leq$	$X + Y > 10$ implies $X > 5$ or $Y > 5$.
(i)	$\leq$	$\min(X, Y) \leq X$ and $\min(X, Y) \leq Y$.
(j)	?	No fixed relation holds in general.
(k)	$\geq$	$E(X^4) \geq (EX^2)^2$.
(l)	$=$	Symmetry, and the two fractions exchange when X, Y are swapped.

For part (k), independence gives

$$E\big(X^2(Y^2+1)\big) = (EX^2)^2 + E(X^2),$$

while the left-hand side is $E(X^4) + E(X^2)$, so the comparison reduces to $E(X^4) \geq (EX^2)^2$.

Final Answer

Part	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
Symbol	$\leq$	$\leq$	$=$	$\leq$	$\leq$	$=$	$\leq$	$\leq$	$\leq$	?	$\geq$	$=$

Problem 8. Ratio of Sample Sums

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 21.

Let $X_1, X_2, \dots$ be i.i.d. positive random variables with mean 2. Let $Y_1, Y_2, \dots$ be i.i.d. positive random variables with mean 3.

Show that

$$\frac{X_1 + X_2 + \cdots + X_n}{Y_1 + Y_2 + \cdots + Y_n} \to \frac{2}{3}$$

with probability 1.

Does it matter whether the X_i are independent of the Y_j?

Solution

By the Strong Law of Large Numbers,

$$\frac{X_1 + \cdots + X_n}{n} \to 2 \qquad \text{with probability 1,}$$

and

$$\frac{Y_1 + \cdots + Y_n}{n} \to 3 \qquad \text{with probability 1.}$$

On the event where both convergences occur, which has probability 1, we may divide the two limits since the denominator tends to the nonzero constant 3. Hence

$$\frac{X_1 + \cdots + X_n}{Y_1 + \cdots + Y_n} = \frac{(X_1 + \cdots + X_n)/n}{(Y_1 + \cdots + Y_n)/n} \to \frac{2}{3}.$$

The independence between the X_i's and the Y_j's is not needed. What is needed is that each sequence separately satisfies the Strong Law. The two probability-one events can be intersected, and the intersection still has probability 1.

Final Answer

$$\frac{X_1 + \cdots + X_n}{Y_1 + \cdots + Y_n} \to \frac{2}{3} \qquad \text{with probability 1.}$$

No cross-independence between the two sequences is needed.

Problem 9. CLT Approximation

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 22.

Let

$$U_1, U_2, \ldots, U_{60}$$

be i.i.d. Unif$(0, 1)$ and let

$$X = U_1 + U_2 + \cdots + U_{60}.$$

(a) Which important distribution is the distribution of X very close to? Specify what the parameters are, and state which theorem justifies your choice.

(b) Give a simple but accurate approximation for

$$P(X > 17).$$

Justify briefly.

Solution

(a) For one Unif$(0, 1)$ random variable,

$$E(U_j) = \frac{1}{2}, \qquad \text{Var}(U_j) = \frac{1}{12}.$$

Therefore

$$E(X) = 60 \cdot \frac{1}{2} = 30, \qquad \text{Var}(X) = 60 \cdot \frac{1}{12} = 5.$$

By the Central Limit Theorem, the distribution of X is very close to

$$N(30, 5).$$

Final Answer for Part (a)

$$X \approx N(30, 5).$$

(b) Using the normal approximation,

$$P(X > 17) \approx P\left(N(30, 5) > 17\right) = P\left(Z > \frac{17 - 30}{\sqrt{5}}\right).$$

The standardized value is

$$\frac{17 - 30}{\sqrt{5}} \approx -5.81.$$

Thus

$$P(X > 17) \approx P(Z > -5.81) = \Phi(5.81),$$

which is essentially 1.

Final Answer for Part (b)

$$P(X > 17) \approx \Phi(5.81) \approx 0.999999997,$$

so a simple accurate approximation is 1.

Problem 10. Exponential Transformation and Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 25.

(a) Let
$$Y = e^X, \qquad X \sim \text{Expo}(3).$$
Find the mean and variance of Y.

(b) For $Y_1, \dots, Y_n$ i.i.d. with the same distribution as Y from part (a), what is the approximate distribution of the sample mean
$$\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$$
when n is large?

Solution

(a) Since $X \sim \text{Expo}(3)$, its MGF is
$$M_X(t) = \frac{3}{3-t}, \qquad t < 3.$$
Because $Y = e^X$,
$$E(Y) = E(e^X) = M_X(1) = \frac{3}{2}.$$
Also
$$E(Y^2) = E(e^{2X}) = M_X(2) = 3.$$
Therefore
$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

Final Answer for Part (a)

$$E(Y) = \frac{3}{2}, \qquad \text{Var}(Y) = \frac{3}{4}.$$

(b) By the Central Limit Theorem, when n is large,
$$\bar{Y}_n \approx N\left(E(Y), \frac{\text{Var}(Y)}{n}\right) = N\left(\frac{3}{2}, \frac{3}{4n}\right).$$

Final Answer for Part (b)

$$\bar{Y}_n \approx N\left(\frac{3}{2}, \frac{3}{4n}\right).$$

Problem 11. MGF of the Poisson Sample Mean

Source: Blitzstein and Hwang, Introduction to Probability, Chapter 10, Problem 27.

Consider i.i.d. Pois(λ) random variables

$$X_1, X_2, \ldots.$$

The MGF of X_j is

$$M(t) = e^{\lambda(e^t - 1)}.$$

(a) Find the MGF $M_n(t)$ of the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j.$$

(b) Find the limit of $M_n(t)$ as $n \to \infty$. You can do this with almost no calculation using a relevant theorem, or you can use part (a) and the fact that $e^x \approx 1 + x$ if x is very small.

Solution

(a) We compute

$$M_n(t) = E\left(e^{t\bar{X}_n}\right) = E\left(e^{(t/n)(X_1 + \cdots + X_n)}\right).$$

By independence,

$$M_n(t) = \prod_{j=1}^{n} E\left(e^{(t/n)X_j}\right) = \left[M\left(\frac{t}{n}\right)\right]^n.$$

Using the Poisson MGF,

$$M_n(t) = \left[e^{\lambda(e^{t/n} - 1)}\right]^n = \exp\left(n\lambda(e^{t/n} - 1)\right).$$

Final Answer for Part (a)

$$M_n(t) = \exp\left(n\lambda(e^{t/n} - 1)\right).$$

(b) As $n \to \infty$,

$$e^{t/n} - 1 \sim \frac{t}{n}.$$

Therefore

$$n\lambda(e^{t/n} - 1) \to \lambda t.$$

It follows that

$$M_n(t) \to e^{\lambda t}.$$

This is the MGF of the constant random variable equal to λ, which agrees with the Law of Large Numbers.

Final Answer for Part (b)

$$\lim_{n\to\infty} M_n(t) = e^{\lambda t}.$$