Chapter 10C: Chi-Square and Student t-Distributions

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- define the Chi-Square distribution;
- understand the connection between Chi-Square and Gamma distributions;
- compute means and variances of Chi-Square random variables;
- understand why Chi-Square distributions appear in statistics;
- define the Student t-distribution;
- understand why t-distributions have heavier tails than Normal distributions;
- understand convergence of the t-distribution to the Normal distribution.

Chi-Square Distribution

Why Chi-Square Distributions Matter

Chi-Square distributions appear throughout statistics:

- sample variance;
- hypothesis testing;
- confidence intervals;
- goodness-of-fit tests;
- contingency tables.

Main idea

Chi-Square distributions measure accumulated squared Normal variation.

Definition of the Chi-Square Distribution

Definition

Suppose

$$Z_1, \ldots, Z_n$$

are independent standard Normal random variables.
Define

$$X = Z_1^2 + \cdots + Z_n^2.$$

Then

$$\boxed{X \sim \chi^2_n.}$$

We say that X has a Chi-Square distribution with

$$n$$

degrees of freedom.

Why Squaring Matters

Each

$$Z_i^2$$

measures squared deviation from the mean.

Adding them gives a total amount of variation:

$$Z_1^2 + \cdots + Z_n^2.$$

Interpretation

Chi-Square random variables measure total squared fluctuation.

Chi-Square and Gamma

Recall: if

$$X \sim \text{Gamma}(\alpha, \lambda),$$

then

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

A Chi-Square distribution is a special Gamma distribution:

$$\boxed{\chi_n^2 = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right).}$$

PDF of the Chi-Square Distribution

Using the Gamma formula:

$$\boxed{f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}, \qquad x > 0.}$$

The shape depends heavily on the degrees of freedom.

Mean and Variance of Chi-Square

If

$$X \sim \chi^2_n,$$

then

$$\boxed{\mathbb{E}(X) = n.}$$

Also,

$$\boxed{\mathrm{Var}(X) = 2n.}$$

Observation

Both the mean and variance grow linearly with the degrees of freedom.

Shape of Chi-Square Distributions

Small degrees of freedom:

- strongly right-skewed;
- concentrated near zero.

Large degrees of freedom:

- less skewed;
- more symmetric;
- approximately Normal.

Connection to CLT

A Chi-Square distribution becomes approximately Normal for large

$$n.$$

Sample Variance and Chi-Square

Suppose

$$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2).$$

Recall the sample variance:

$$S_n^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2.$$

Then:

$$\boxed{\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}.}$$

This is one of the most important results in classical statistics.

Degrees of Freedom

Why do we get

$$n - 1$$

degrees of freedom instead of

$$n?$$

Because:

$$\bar{X}_n$$

has already been estimated from the data.

Once the sample mean is fixed, only

$$n - 1$$

deviations can vary independently.

Chi-Square Additivity

Theorem

If

$$X \sim \chi^2_m, \qquad Y \sim \chi^2_n,$$

independently, then

$$\boxed{X + Y \sim \chi^2_{m+n}.}$$

This follows immediately from the Gamma representation.

Student t-Distribution

Why the t-Distribution?

Suppose:

- the population is Normal;
- the variance

  σ^2

  is unknown;
- the sample size is small.

Then the standard Normal approximation is no longer exact.

Solution

The Student t-distribution accounts for uncertainty in estimating the variance.

Definition of the Student t-Distribution

Definition

Let

$$Z \sim N(0,1),$$

and

$$V \sim \chi^2_n,$$

independently.
Define

$$\boxed{T = \frac{Z}{\sqrt{V/n}}.}$$

Then

$$T$$

has the Student t-distribution with

$$n$$

degrees of freedom.

Why the t-Distribution Has Heavy Tails

The denominator:

$$\sqrt{V/n}$$

is random.

Sometimes it becomes unusually small.

That makes:

$$|T|$$

occasionally very large.

Consequence

The t-distribution has heavier tails than the Normal distribution.

Properties of the t-Distribution

The t-distribution is:

- symmetric about

  0;
- bell-shaped;
- heavier-tailed than the Normal distribution.

As the degrees of freedom increase:

- tails become lighter;
- the distribution approaches the Normal distribution.

As

$$n \to \infty,$$

the Chi-Square random variable satisfies:

$$\frac{V}{n} \to 1.$$

Therefore:

$$T = \frac{Z}{\sqrt{V/n}} \approx Z.$$

Thus:

$$\boxed{t_n \to N(0,1).}$$

The Cauchy Distribution

If

$$n = 1,$$

then the t-distribution becomes the Cauchy distribution.

The Cauchy distribution has:

- extremely heavy tails;
- undefined mean;
- undefined variance.

Important

Not every distribution has finite expectation or variance.

Confidence Intervals

The t-distribution is fundamental for confidence intervals.

For example:

$$\bar{X}_n \pm t^* \frac{S_n}{\sqrt{n}}.$$

Here:

- $$\bar{X}_n$$
  is the sample mean;
- $$S_n$$
  estimates the standard deviation;
- $$t^*$$
  comes from the t-distribution.

Normal vs t-Distribution

Distribution	Tails	Variance known?
Normal	lighter	yes
Student t	heavier	no

Interpretation

The heavier tails account for uncertainty in estimating the variance.

Big Picture

Three major distributions now fit together:

Distribution	Built From	Main Role
Normal	sums/CLT	averages
Chi-Square	squared Normals	variances
Student t	Normal divided by Chi-Square	inference

Summary

- Chi-Square distributions are sums of squared standard Normals.
- Chi-Square distributions are special Gamma distributions.
- Sample variances naturally produce Chi-Square distributions.
- Student t-distributions arise when variance is estimated.
- t-distributions have heavier tails than Normals.
- As degrees of freedom increase:

$$t_n \to N(0, 1).$$

End of Probability I

The LLN, CLT, Chi-Square, and t-distributions form the mathematical foundation of classical statistics.