Chapter 1A: Why Probability? Sample Spaces and Events

MATH/STAT 394: Probability I

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Summer 2026

Learning Goals

By the end of this lecture, students should be able to:

- explain probability as a mathematical language for uncertainty;
- identify a sample space for a random experiment;
- distinguish between outcomes and events;
- translate basic probability statements into set notation;
- use unions, intersections, complements, and De Morgan's laws;

Why Study Probability?

Central idea

Probability is the logic of uncertainty.

Probability provides tools for:

- modeling randomness and uncertainty;
- understanding variation and separating signal from noise (blood pressure)
- making principled predictions and decisions;
- quantifying risk;
- avoiding misleading or incomplete intuition.

Three Habits for This Course

1. **Biohazards.** Study common mistakes so that invalid reasoning becomes easier to detect.
2. **Sanity checks.** After solving a problem, check whether the answer makes sense in simple or extreme cases.
3. **Conceptual clarity.** Make sure the sample space, events, and assumptions are clearly specified before assigning probabilities.

Guiding principle

Good probability reasoning starts with precise modeling, not just calculations.

Sample Spaces and Events

Random Experiments

A **random experiment** is a process whose outcome is uncertain before it is performed.

Examples:

- flipping a coin;
- rolling a die;
- drawing a card from a shuffled deck;
- recording tomorrow's temperature;
- selecting a student from the class;
- measuring the lifetime of a machine component.

Important

Before assigning probabilities, the possible outcomes must be clearly specified.

Definition: Sample Space and Event

Definition

The **sample space** $\mathcal{S}$ of an experiment is the set of all possible outcomes.
An **event** A is a subset of the sample space:

$$A \subseteq \mathcal{S}.$$

The event A occurs if the actual outcome belongs to A:

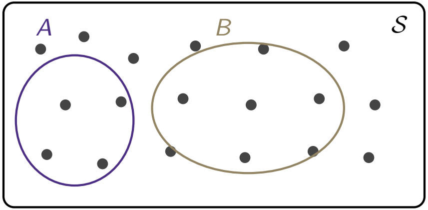
$$s_{\text{actual}} \in A.$$

- An **outcome** is a single element of $\mathcal{S}$.
- An **event** is a collection of outcomes.
- The sample space is the universal set for the experiment.
- Some textbooks use Ω instead of $\mathcal{S}$, and ω instead of s

Pebble World Picture

When the sample space is finite, imagine each possible outcome as a pebble.

$$\mathcal{S} = \{\text{all pebbles}\}$$



- Each pebble represents an outcome.
- An event is a set of pebbles.
- Performing the experiment selects one pebble.
- The event occurs if the selected pebble lies in the event.

Types of Sample Spaces

Sample spaces can be:

1. **Finite**
$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$
for rolling one die.
2. **Countably infinite**
$$\mathcal{S} = \{1, 2, 3, \dots\}$$
for the number of tosses until the first head.
3. **Uncountable**
$$\mathcal{S} = [0, \infty)$$
for the lifetime of a machine component.

For now

Many early examples use finite sample spaces, where outcomes can be counted.

Set Operations for Events

Union, Intersection, Complement

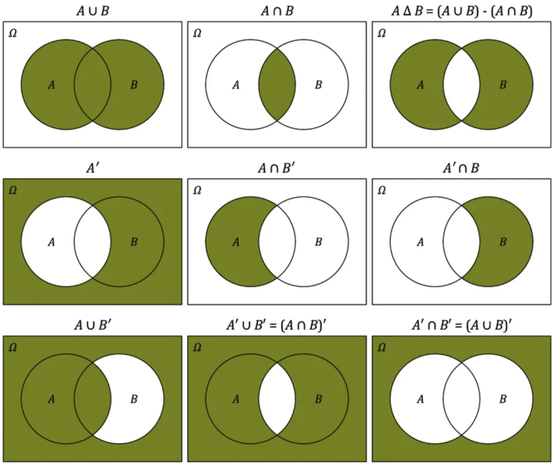
Let $A, B \subseteq \mathcal{S}$ be events.

Set operation	Notation	Meaning
Union	$A \cup B$	A or B occurs, possibly both
Intersection	$A \cap B$	both A and B occur
Complement	A^c (or A')	A does not occur

Notation

Some textbooks write A' for the complement of A. In this course, we write A^c instead, because prime notation is also used for differentiation, such as $f'(x)$.

Venn Diagrams for Basic Set Operations



The green area represents the formula above each diagram. Source: ResearchGate Figure

Here Ω represent the sample space ($\mathcal{S}$) and A' represent complement of A, i.e., A^c .

Example: Ten Coin Flips

Suppose a coin is flipped 10 times.

$$\mathcal{S} = \{(s_1, \ldots, s_{10}) : s_j \in \{H, T\},\ 1 \le j \le 10\}.$$

There are

$$|\mathcal{S}| = 2^{10}$$

possible outcomes.
Let A_j be the event that the jth flip is heads:

$$A_j = \{(s_1, \ldots, s_{10}) \in \mathcal{S} : s_j = H\}.$$

$$\text{At least one head} = \bigcup_{j=1}^{10} A_j.$$

$$\text{All flips are heads} = \bigcap_{j=1}^{10} A_j.$$

$$\text{At least two consecutive heads} = \bigcup_{j=1}^{9} (A_j \cap A_{j+1}).$$

The Empty Set

The **empty set** is the set with no elements. It is denoted by

$$\emptyset.$$

Important facts:

- $\emptyset$ means "nothing occurred" or "no outcome is included."
- For every set A,
$$A \cap \emptyset = \emptyset.$$
- For every set A,
$$A \cup \emptyset = A.$$
- The empty set is a subset of every set:
$$\emptyset \subseteq A.$$

Important Set Notation Clarifications

Singletons versus elements

The object 2 is not the same as the set $\{2\}$.

$$2 \neq \{2\}.$$

The set $\{2\}$ is the set whose only element is 2.

Similarly,

$$\emptyset \neq \{\emptyset\}.$$

- $\emptyset$ has no elements.
- $\{\emptyset\}$ has one element, namely $\emptyset$.

So:

$$|\emptyset| = 0, \qquad |\{\emptyset\}| = 1.$$

Events as Sets: A versus $\{A\}$

Suppose A is an event. Then A is already a set of outcomes.
For example, if

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}, \qquad A = \{2, 4, 6\},$$

then A is the event that the die roll is even.

But

$$\{A\} = \{\{2, 4, 6\}\}$$

is a set whose single element is the set A.

Common mistake

Usually, $A \subseteq \mathcal{S}$, but $\{A\}$ is not a subset of $\mathcal{S}$ unless A itself is an outcome.

De Morgan's Laws

For events $A, B \subseteq \mathcal{S}$,

$$(A \cup B)^c = A^c \cap B^c,$$

$$(A \cap B)^c = A^c \cup B^c.$$

Interpretation:

- Not having at least one of A or B means having neither.
- Not having both A and B means at least one fails to occur.

Optional live verification

Draw a Venn diagram and shade both sides of each identity.

Union and Intersection of Many Events

Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{S}$.
The union is the event that **at least one** of the events occurs:

$$\bigcup_{i=1}^{n} A_i = \{s \in \mathcal{S} : s \in A_i \text{ for at least one } i \in \{1, \ldots, n\}\}.$$

The intersection is the event that **all** of the events occur:

$$\bigcap_{i=1}^{n} A_i = \{s \in \mathcal{S} : s \in A_i \text{ for every } i \in \{1, \ldots, n\}\}.$$

For infinitely many events $A_1, A_2, \ldots$:

$$\bigcup_{i=1}^{\infty} A_i = \{s \in \mathcal{S} : s \in A_i \text{ for at least one } i\}, \qquad \bigcap_{i=1}^{\infty} A_i = \{s \in \mathcal{S} : s \in A_i \text{ for every } i\}.$$

Question: What does demorgan law generalize to, for a union of (infinitely many) events?

Example: Drawing One Card

Draw one card from a standard 52-card deck.
Define:

$$A = \{\text{card is an ace}\}, \qquad B = \{\text{card has a black suit}\},$$
$$D = \{\text{card is a diamond}\}, \qquad H = \{\text{card is a heart}\}.$$

Then:

$$A \cap H = \{\text{Ace of Hearts}\},$$

$$A \cap B = \{\text{Ace of Spades, Ace of Clubs}\},$$

$$D \cap H = \emptyset,$$

because no card can be both a diamond and a heart.

$$A \cup D \cup H = \{\text{card is red or an ace}\}.$$

Card Example: Complements

Using De Morgan's laws,

$$(A \cup B)^c = A^c \cap B^c.$$

So:

$$(A \cup B)^c = \{\text{card is not an ace and not black}\}.$$

Equivalently,

$$(A \cup B)^c = \{\text{card is a red non-ace}\}.$$

Also,

$$(D \cup H)^c = D^c \cap H^c = B.$$

Lesson

The same event can often be expressed in more than one way. Some expressions are much easier to use than others.

A Subtle Point: Is the Sample Space Correct?

In the card example, the sample space is the set of all 52 standard cards.

If the drawn card is a joker, then the issue is not probability calculation.

The issue is:

the sample space was wrong.

Rule

The sample space must contain every outcome that can actually occur.

English $\longleftrightarrow$ Set Notation

Dictionary: Events and Occurrences

Let s_{actual} be the actual outcome.

English statement	Set notation
sample space	$\mathcal{S}$
s is a possible outcome	$s \in \mathcal{S}$
A is an event	$A \subseteq \mathcal{S}$
A occurred	$s_{\text{actual}} \in A$
something must happen	$s_{\text{actual}} \in \mathcal{S}$

Conceptual point

Events are not numbers. Events are sets. Probability will later assign numbers to events.

Dictionary: New Events from Old Events

English statement	Set notation
A or B	$A \cup B$
A and B	$A \cap B$
not A	A^c
A or B, but not both	$(A \cap B^c) \cup (A^c \cap B)$
at least one of $A_1, \ldots, A_n$	$\bigcup_{j=1}^{n} A_j$
all of $A_1, \ldots, A_n$	$\bigcap_{j=1}^{n} A_j$

Notation reminder

We write A^c for complement, not A'.

Dictionary: Relationships Between Events

English statement	Set notation
A implies B	$A \subseteq B$
A and B are mutually exclusive	$A \cap B = \emptyset$
$A_1, \ldots, A_n$ form a partition of $\mathcal{S}$	$A_1 \cup \cdots \cup A_n = \mathcal{S}$, $A_i \cap A_j = \emptyset$ for $i \neq j$

Partition

A partition divides the sample space into non-overlapping pieces that together cover all possible outcomes.

Mini-Exercise: Translate to Set Notation

Let A_j be the event that the jth coin flip is heads.
Translate each statement:

1. The first flip is tails.
2. The first two flips are heads.
3. At least one of the first three flips is heads.
4. None of the first three flips is heads.
5. Exactly one of the first two flips is heads.

Optional answers

$$A_1^c, \qquad A_1 \cap A_2, \qquad A_1 \cup A_2 \cup A_3,$$
$$(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c,$$
$$(A_1 \cap A_2^c) \cup (A_1^c \cap A_2).$$

Common Mistake

Biohazard

Do not confuse an outcome with an event.

For one die roll:

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

The number 4 is an outcome.
The event "the roll is even" is the set

$$E = \{2, 4, 6\}.$$

So:

$$4 \in \mathcal{S}, \qquad E \subseteq \mathcal{S}.$$

Emphasis

Writing $E \in \mathcal{S}$ is wrong here because E is a set of outcomes, not a single outcome.

Summary

- Probability begins with a random experiment, and then, a clearly specified sample space $\mathcal{S}$.
- Outcomes are elements of $\mathcal{S}$ (i.e., $s \in S$).
- Events are subsets of $\mathcal{S}$ (i.e., $A \subseteq S$).
- Set operations: allow us to build complicated events from simpler events.
- De Morgan's laws: essential tools for rewriting complements.
- Choosing the correct sample space is part of the modeling process.

Next lecture

Naive probability: when outcomes are equally likely, probability becomes counting.