Chapter 1B: Naive Probability

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- state the naive definition of probability;
- identify when the naive definition is appropriate;
- compute probabilities by counting favorable outcomes;
- use complements to simplify probability calculations;

Naive Probability

From Events to Probabilities

In the previous lecture:

$$\mathcal{S} = \{\text{all possible outcomes}\}, \qquad A \subseteq \mathcal{S}.$$

Now we want to assign a number to an event:

$$\mathbb{P}(A) = \text{probability that } A \text{ occurs.}$$

Question

If every outcome in a finite sample space is equally likely, how should we define $\mathbb{P}(A)$?

Definition: Naive Probability

Naive Definition of Probability

Let A be an event for an experiment with a finite sample space $\mathcal{S}$.
If all outcomes in $\mathcal{S}$ are equally likely, define

$$\mathbb{P}_{\text{naive}}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{\text{number of outcomes favorable to } A}{\text{total number of outcomes in } \mathcal{S}}.$$

Here $|A|$ denotes the number of elements in the set A (aka $card(A)$).

Question: Why does this definition is suitable for equally-likely outcomes?

Naive Probability - Properties

Naive probability satisfies

$$0 \leq \mathbb{P}(A) \leq 1,\ \mathbb{P}(S) = 1,\ \mathbb{P}(\emptyset) = 0, \qquad \text{for every } A \subseteq \mathcal{S}.$$

Why? First, if $A \subseteq \mathcal{S}$, then

$$0 \leq |A| \leq |\mathcal{S}|,$$

so

$$0 \leq \frac{|A|}{|\mathcal{S}|} \leq 1.$$

Also,

$$\mathbb{P}(\mathcal{S}) = \frac{|\mathcal{S}|}{|\mathcal{S}|} = 1, \qquad \mathbb{P}(\emptyset) = \frac{0}{|\mathcal{S}|} = 0.$$

Additivity for Disjoint Events

Suppose $A, B \subseteq \mathcal{S}$,
Then, since

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Therefore,

$$\mathbb{P}(A \cup B) = \frac{|A \cup B|}{|\mathcal{S}|} = \frac{|A| + |B| - |A \cap B|}{|\mathcal{S}|} = \frac{|A|}{|\mathcal{S}|} + \frac{|B|}{|\mathcal{S}|} - \frac{|A \cap B|}{|\mathcal{S}|}.$$

Thus

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

A speical case is when A and B are exclusive ($A \cap B = \emptyset$), when we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Looking ahead

Later, we will define probability more generally using properties like these as axioms. Naive probability will then become a special case of the general theory.

What the Definition Means

The naive definition says:

$$\mathbb{P}(A) = \frac{\text{how many outcomes make } A \text{ happen}}{\text{how many outcomes are possible}}.$$

This requires two assumptions:

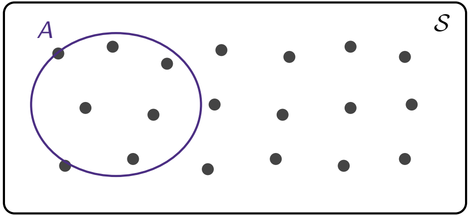
1. The sample space $\mathcal{S}$ is finite (we can't divide by infinity).
2. All outcomes in $\mathcal{S}$ are equally likely.

Important

The naive definition is not wrong. It is just limited. It should only be used when its assumptions are justified.

Suppose each pebble has the same mass.

$$\mathbb{P}(A) = \frac{\text{number of pebbles in } A}{\text{number of pebbles in } \mathcal{S}}.$$



$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|}.$$

Example: One Fair Die

Roll one fair six-sided die.

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

Let

$$A = \{\text{roll is even}\} = \{2, 4, 6\}.$$

Then

$$|\mathcal{S}| = 6, \qquad |A| = 3.$$

Therefore

$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{3}{6} = \frac{1}{2}.$$

Example: One Card

Draw one card from a standard well-shuffled 52-card deck.
Let

$$A = \{\text{card is an ace}\}.$$

There are 4 aces and 52 cards total, so

$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|} = \frac{4}{52} = \frac{1}{13}.$$

Let

$$H = \{\text{card is a heart}\}.$$

Then

$$\mathbb{P}(H) = \frac{13}{52} = \frac{1}{4}.$$

Example: Red or Ace

Draw one card from a standard 52-card deck.
Let

$$R = \{\text{card is red}\}, \qquad A = \{\text{card is an ace}\}.$$

We want

$$\mathbb{P}(R \cup A).$$

There are:

$$26 \text{ red cards}, \qquad 4 \text{ aces}, \qquad 2 \text{ red aces}.$$

So

$$|R \cup A| = 26 + 4 - 2 = 28.$$

Therefore

$$\mathbb{P}(R \cup A) = \frac{28}{52} = \frac{7}{13}.$$

Complements

Complement Rule Under the Naive Definition

Let $A \subseteq \mathcal{S}$.
The complement of A is

$$A^c = \{s \in \mathcal{S} : s \notin A\}.$$

Since every outcome is either in A or not in A,

$$|A^c| = |\mathcal{S}| - |A|.$$

Therefore,

$$\mathbb{P}(A^c) = \frac{|A^c|}{|\mathcal{S}|} = \frac{|\mathcal{S}| - |A|}{|\mathcal{S}|} = 1 - \frac{|A|}{|\mathcal{S}|}.$$

So

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Why Complements Are Useful

Sometimes it is easier to count when an event does **not** happen.

Instead of computing $\mathbb{P}(A)$ directly, compute:

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c).$$

Strategy

When a probability asks for phrases like

"at least one", "at least two", "not all",

first ask whether the complement is easier.

Example: At Least One Head

Flip a fair coin 5 times.
Let

$$A = \{\text{at least one head occurs}\}.$$

The complement is

$$A^c = \{\text{no heads occur}\}.$$

There is only one outcome with no heads: $TTTTT$.
The total number of outcomes is $2^5 = 32$.
Therefore

$$\mathbb{P}(A^c) = \frac{1}{32}, \qquad \mathbb{P}(A) = 1 - \frac{1}{32} = \frac{31}{32}.$$

Example: At Least One Six

Roll a fair die 4 times.
Let

$$A = \{\text{at least one six occurs}\}.$$

It is easier to count

$$A^c = \{\text{no six occurs}\}.$$

Total outcomes: 6^4. Outcomes with no six: 5^4.
Therefore,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{5^4}{6^4}.$$

Common Mistakes

Example: Two Fair Dice

Roll two fair dice.
A correct sample space is

$$\mathcal{S} = \{(i, j) : i, j \in \{1, 2, 3, 4, 5, 6\}\}.$$

Here (i, j) means:

$$i = \text{result of die 1}, \qquad j = \text{result of die 2}.$$

Thus

$$|\mathcal{S}| = 36.$$

Important

The outcomes are ordered pairs. For example, $(5, 6)$ and $(6, 5)$ are different outcomes.

Example: Sum 11 versus Sum 12

For two fair dice:

$$\mathcal{S} = \{(i, j) : i, j \in \{1, \ldots, 6\}\}.$$

Let

$$A = \{\text{sum is } 11\}, \qquad B = \{\text{sum is } 12\}.$$

Then

$$A = \{(5, 6), (6, 5)\}, \qquad B = \{(6, 6)\}.$$

So

$$\mathbb{P}(A) = \frac{2}{36} = \frac{1}{18},$$

while $\mathbb{P}(B) = \frac{1}{36}$.
Thus a sum of 11 is twice as likely as a sum of 12.

Mini-Exercises

1. A fair die is rolled. What is the probability of rolling a number at least 5?
2. A fair coin is flipped 4 times. What is the probability of getting all tails?
3. A card is drawn from a standard deck. What is the probability that it is not a heart?
4. Two fair dice are rolled. What is the probability that the sum is 7?
5. Why is the statement "the probability of rain tomorrow is 1/2 because either it rains or it does not" invalid?

Answers

$$\frac{2}{6} = \frac{1}{3}, \qquad \frac{1}{16}, \qquad 1 - \frac{13}{52} = \frac{3}{4}, \qquad \frac{6}{36} = \frac{1}{6}.$$

Summary

- The naive definition is
$$\mathbb{P}(A) = \frac{|A|}{|\mathcal{S}|}.$$
- It applies when $\mathcal{S}$ is finite and all outcomes are equally likely.
- Naive probability satisfies:
$$0 \leq \mathbb{P}(A) \leq 1, \qquad \mathbb{P}(\mathcal{S}) = 1, \qquad \mathbb{P}(\emptyset) = 0,$$
If $A \cap B = \emptyset$, then
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$
- Complements are often useful:
$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Next lecture

Counting methods: multiplication rule, sampling with replacement, and sampling without replacement.