Chapter 1C–1D: Counting, Multiplication Rule, and Binomial Coefficients

MATH/STAT 394: Probability I

Arman Jahangiri

University of Washington Department of Mathematics

Summer 2026

Learning Goals

By the end of this lecture, students should be able to:

- apply the multiplication rule for counting;
- distinguish between sampling with and without replacement;
- compute permutations and ordered selections;
- compute combinations and unordered selections;
- understand the binomial coefficient combinatorially;
- recognize and avoid overcounting;
- prove combinatorial identities using counting arguments.

The Multiplication Rule (Product Rule)

Multiplication Rule

Suppose a procedure consists of two stages.

- Stage 1 can be done in **a** ways;
- for each choice in Stage 1, Stage 2 can be done in **b** ways.

Then the entire procedure can be done in

$$\mathbf{a} \times \mathbf{b}$$

ways.

More generally, if a process has stages with

$$n_1, n_2, \ldots, n_k$$

possible choices respectively, then the total number of outcomes is $n_1 n_2 \cdots n_k$.

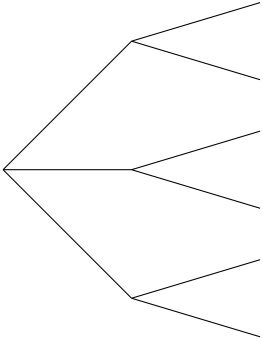
Tree Interpretation

The multiplication rule can be visualized using a tree.

Suppose:

- there are 3 choices for the first step;
- each leads to 2 choices for the second step.

Then, according to the Multiplication Rule, there are $3 \times 2 = 6$ possible outcomes.



Suppose a license plate consists of 3 letters, followed by 4 digits. Assume repetition is allowed.

Figure: WA license plate. Source: Wikipedia.

Choices:

26 possibilities for each letter,

10 possibilities for each digit.

Thus the total number of possible license plates is

$$(26 \times 26 \times 26) \times (10 \times 10 \times 10 \times 10) = 26^3 \cdot 10^4 = 175,760,000 \quad (?!)$$

Example: Binary Strings

A binary string of length n consists only of 0's and 1's.

Each position has:

$$2 \text{ choices.}$$

Using the multiplication rule:

$$\underbrace{\underset{0 \text{ or } 1}{\boxed{2}}}_{\text{position } 1} \times \underbrace{\underset{0or1}{\boxed{2}}}_{\text{position } 2} \times \cdots \times \underbrace{\underset{0or1}{\boxed{2}}}_{\text{position } n} = 2^n.$$

Therefore, Number of binary strings of length n is 2^n.
Remark: This is also equal to the number of all possible subsets of a set with n elements. (Why?)

Sampling With and Without Replacement

Ordered Sampling With Replacement

Suppose we have a set of size n, and we perform k draws with replacement, where $1 \leq k \leq n$.

With replacement means: after each draw, the selected object is returned before the next draw.

Therefore, each draw has exactly n possible choices.

Using the multiplication rule:

$$\underbrace{\begin{array}{c}\boxed{n}\\ \text{choices}\end{array}}_{\text{draw } 1} \times \underbrace{\begin{array}{c}\boxed{n}\\ \text{choices}\end{array}}_{\text{draw } 2} \times \cdots \times \underbrace{\begin{array}{c}\boxed{n}\\ \text{choices}\end{array}}_{\text{draw } k} = n^k.$$

Hence, the number of possible ways to draw k ordered items, with replacement, from a set of size n, is

$$n^k.$$

Example: PIN Codes

A 4-digit PIN code uses digits 0–9.
Digits may repeat.

Each position has:

$$10 \text{ choices.}$$

Therefore:

$$10^4 = 10000$$

possible PINs.

Ordered Sampling Without Replacement

Now suppose selected objects are *not* returned after each draw.

Then the number of available choices decreases after every selection.

Using the multiplication rule:

$$\underbrace{\underset{\text{choices}}{\boxed{n}}}_{\text{draw } 1} \times \underbrace{\underset{\text{choices}}{\boxed{n-1}}}_{\text{draw } 2} \times \underbrace{\underset{\text{choices}}{\boxed{n-2}}}_{\text{draw } 3} \times \cdots \times \underbrace{\underset{\text{choices}}{\boxed{n-k+1}}}_{\text{draw } k}$$

$$= n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Important

Without replacement means that the choices depend on previous selections.

Example: Arranging Books

Suppose 5 distinct books are arranged on a shelf.

Choices:

$$5 \text{ choices for the first position,}$$

$$4 \text{ for the second,}$$

$$3 \text{ for the third,}$$

etc.
Therefore, the number of arrangements is

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120.$$

Factorials and Permutations

Factorials

Definition

For a positive integer n,

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

Also define

$$0! = 1.$$

Examples:

$$1! = 1, \qquad 2! = 2, \qquad 3! = 6, \qquad 4! = 24, \qquad 5! = 120.$$

Permutations

A **permutation** is an ordered arrangement.

The number of ordered ways to choose k objects from n distinct objects is

$$n(n-1)\cdots(n-k+1).$$

Using factorial notation:

$$\frac{n!}{(n-k)!}$$

This is denoted by

$$P(n,k).$$

So, in summary

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1).$$

Example: Choosing Officers

Suppose a club has 12 members.
We choose:

- a president,
- a vice president,
- a secretary.

Since positions are different, order matters.

Therefore:

$$P(12, 3) = \frac{12!}{9!} = 12 \cdot 11 \cdot 10 = 1320.$$

Combinations

When Order Does Not Matter

Suppose we choose a committee of size k from n people.

Now:

$$\{A, B, C\}$$

and

$$\{B, A, C\}$$

represent the same committee.

Thus order does **not** matter.

This requires a different counting formula.

Definition

The number of subsets of size k from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This is read as:

"n choose k".

Note: $\binom{1}{1} = \binom{1}{0} = \binom{0}{0} = 1$, can be justified by the formula or combinatorially.

Why Divide by $k!$?

Suppose we first count ordered selections:

$$P(n,k) = \frac{n!}{(n-k)!}.$$

But each unordered subset is counted many times.

For example $\{A, B, C\}$ can appear as:

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA.$$

There are $3! = 6$ orderings.
Therefore we divide by $k!$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!} \Rightarrow \binom{n}{k} \le P(n,k).$$

Example: Poker Hands

A poker hand consists of 5 cards from a 52-card deck.
Order does not matter.

Thus the number of poker hands is

$$\binom{52}{5} = \frac{52!}{5!47!}.$$

Example: Committees

How many committees of size 4 can be formed from 10 people?

Since order does not matter:

$$\binom{10}{4} = \frac{10!}{4!6!}.$$

Simplifying:

$$\binom{10}{4} = 210.$$

Properties of Binomial Coefficients

Symmetry Identity

Identity

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

Reason:
Choosing k elements to include is equivalent to choosing $n-k$ elements to exclude. :)
Algebraic Proof:

Khayyam–Pascal's Identity

Khayyam-Pascal's Identity

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

Combinatorial Proof of Khayyam-Pascal's Identity

Suppose a committee of size k is selected from n people.
Focus on one distinguished person.
Two possibilities:

1. The distinguished person is not selected.
   Then choose all k members from the remaining $n-1$ people:
$$\binom{n-1}{k}.$$
2. The distinguished person is selected.
   Then choose the remaining $k-1$ members from the remaining $n-1$ people:
$$\binom{n-1}{k-1}.$$

Adding the two cases gives (why addition?)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Khayyam–Pascal Triangle: The Main Idea

The Khayyam–Pascal triangle is a triangular arrangement of numbers.

Each row begins and ends with 1.

Every interior entry is obtained by adding the two entries directly above it:

$$\text{new entry} = \text{upper-left entry} + \text{upper-right entry}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Source: Wikipedia: Pascal's Triangle.

Building the Triangle

Start with 1 at the top.

Each new number is the sum of the two numbers above it.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & \boxed{1+1} & & 1 & & \\
 & 1 & & \boxed{1+2} & & \boxed{2+1} & & 1 & \\
1 & & \boxed{1+3} & & \boxed{3+3} & & \boxed{3+1} & & 1
\end{array}
$$

The Formula Inside the Triangle

$$
\begin{array}{lccccccccccccc}
\text{row } 0 & & & & & & & \binom{0}{0} & & & & & & \\
\text{row } 1 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
\text{row } 2 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
\text{row } 3 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & & & & & & \vdots & & & & & & \\
\text{row } n-1 & & \binom{n-1}{0} & \cdots & & \binom{n-1}{k-1} & & \binom{n-1}{k} & & \cdots & & \binom{n-1}{n-1} & & \\
\text{row } n & \binom{n}{0} & & \cdots & \binom{n}{k-1} & & \binom{n}{k} & & \binom{n}{k+1} & & \cdots & & \binom{n}{n} &
\end{array}
$$

Each interior entry is obtained by adding the two entries directly above it:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

This identity is known as **Khayyam-Pascal's Rule**.

Khayyam-Pascal's Rule/Identity

The rule for building the triangle is

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $1 \leq k \leq n-1$.

For example,

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6.$$

So every interior entry is obtained by adding the two entries above it.

Connection to Binomial Expansions

The rows of the triangle give the coefficients in binomial expansions.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

- The coefficients are exactly the rows of the Khayyam–Pascal triangle.
- Hence, we can show that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ (a very useful identity in Probability theory.
- Also proves why $2^n = \sum_{i=0}^{n} \binom{n}{k}$.
  Hint: $2 = 1 + 1$:)

Story Proofs and Overcounting

A **story proof** proves an identity by showing that both sides count the same object in different ways.

Advantages:

- gives intuition;
- explains why identities are true;
- often simpler than algebraic proofs.

Example Story Proof

Identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Why?

The left-hand side counts subsets of an n-element set by grouping according to subset size.

The right-hand side counts subsets directly (see slide 60):
each element is either:

$$\text{included} \quad \text{or} \quad \text{not included.}$$

Thus there are

$$2^n$$

subsets total.

Mini-Exercises

1. How many binary strings of length 8 are there?
2. How many 3-letter strings can be formed from the English alphabet if repetition is allowed?
3. How many ordered arrangements of 4 people can be selected from 10 people?
4. How many committees of size 3 can be formed from 8 people?

Answers

$$2^8 = 256, \qquad 26^3, \qquad \frac{10!}{6!}, \qquad \binom{8}{3} = 56.$$

Summary

- Number of ordered length-k sequences from n options (repetition allowed):
$$n^k.$$
- Number of ordered length-k sequences from n distinct objects (no repetition):
$$\frac{n!}{(n-k)!}.$$
- Number of permutations of k objects chosen from n:
$$P(n,k) = \frac{n!}{(n-k)!}.$$
- Number of combinations of k objects chosen from n:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
- Fundamental identities for binomial coefficients:
$$\binom{n}{k} = \binom{n}{n-k},$$
and
$$\binom{n}{} = \binom{n-1}{} + \binom{n-1}{}$$