Chapter 2: Conditional Probability

MATH/STAT 394: Probability I

Arman Jahangiri

University of Washington Department of Mathematics

Summer 2026

Learning Goals

By the end of this lecture, students should be able to:

- interpret conditional probability intuitively;
- apply Bayes' rule and the Law of Total Probability;
- distinguish between $P(A|B)$ and $P(B|A)$;
- understand independence and conditional independence;
- use conditioning as a strategy in solving in probability problems.

Conditional Probability

Why Do We Need Conditional Probability?

In many situations, probabilities change once additional information becomes available.

Before observing new information

Let
$$R = \{\text{it rains today}\}.$$
Without any additional information, we assign the probability
$$\mathbb{P}(R).$$
This represents our initial uncertainty about rain.

After observing new information

Suppose we now observe dark clouds:
$$C = \{\text{dark clouds are present}\}.$$
The probability of rain may now change to
$$\mathbb{P}(R \mid C),$$
read as "probability of R given C.".

An important observation here is that we do not necessarily have $\mathbb{P}(R \mid C) = \mathbb{P}(R)$, because the information that C occurred changes what outcomes now appear more plausible.

Main Idea

Conditional probability quantifies how probabilities are updated after new information or evidence is observed.

Definition of Conditional Probability

Definition

If

$$\mathbb{P}(B) > 0,$$

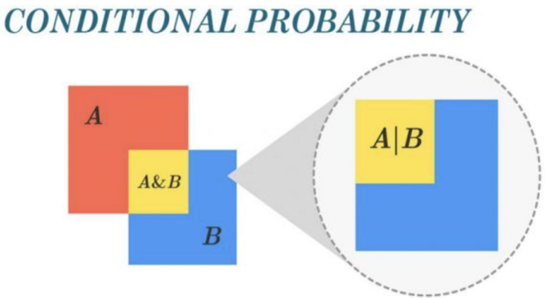
then the conditional probability of A given B is

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.}$$

Interpretation:

- B is the observed evidence;
- we restrict attention to outcomes in B;
- probabilities are renormalized within B.

Visual Interpretation of Conditional Probability



When conditioning on B, the event B becomes the new sample space (i.e., replaces $\mathcal{S}$). The conditional probability $\mathbb{P}(A \mid B)$ measures the proportion of B that also belongs to A.

$$\boxed{\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}} \qquad \text{provided } \mathbb{P}(B) > 0.$$

Image adapted from: ExcelR — Conditional Probability.

Useful Equations from Conditional Probability

$$\mathbb{P}(A) = \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c).$$

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c).$$

Law of Total Probability

Each equation partitions the sample space into two disjoint cases and adds the corresponding conditional probabilities.

Example: Two Cards

Two cards are drawn without replacement.
Define

$$A = \{\text{first card is a heart}\}, \qquad B = \{\text{second card is red}\}.$$

Using the multiplication rule,

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\mathbb{P}(A) = \frac{25}{51} \cdot \frac{13}{52} = \frac{25}{204}.$$

Also,

$$\mathbb{P}(A) = \frac{1}{4},$$

and

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c) = \frac{25}{51} \cdot \frac{1}{4} + \frac{17}{34} \cdot \frac{3}{4} = \frac{1}{2}.$$

Therefore,

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{25/204}{1/4} = \frac{25}{51} \approx 0.490,$$

and

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{25/204}{1/2} = \frac{25}{102} \approx 0.245.$$

Important Warning

Conditional probabilities are directional

In general,

$$\boxed{\mathbb{P}(A|B) \neq \mathbb{P}(B|A).}$$

The conditioning event matters.

$$\mathbb{P}(A|B)$$

means:

$$\text{probability of } A \text{ after learning } B.$$

Bayes' Rule and LOTP

From the definition:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Multiplying both sides by $\mathbb{P}(B)$ gives:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B).$$

Similarly,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A).$$

The last two equalities together imply an important equality in Probability, known as the Bayes' rule, given in the next slide.

Bayes' Rule: Updating Beliefs with Evidence

Bayes' Rule

$$\boxed{\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}.}$$

Bayes' rule describes how probabilities "evolve" when new information is observed.

$$\underbrace{\mathbb{P}(A)}_{\text{before observing } B} \implies \underbrace{\mathbb{P}(A \mid B)}_{\text{after observing } B} = \mathbb{P}(A) \times \underbrace{\frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)}}_{\text{updating weight}}$$

The observation of B changes our belief about whether A occurred.

Interpretation

Bayes' rule updates a prior belief $\mathbb{P}(A)$ into a posterior belief $\mathbb{P}(A \mid B)$ by multiplying by a factor that measures how strongly the evidence B supports A.

Law of Total Probability

Suppose

$$A_1, \ldots, A_n$$

form a partition of $\mathcal{S}$.
Then:

$$\boxed{\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i).}$$

Interpretation

Break the sample space into cases, compute probabilities within each case, then combine them.

Rare Disease Example

Suppose we are analyzing the disease rate in a certain area.
Let

$$D = \{\text{person has the disease}\}, \quad T = \{\text{test result is positive}\},$$

and assume

$$\mathbb{P}(D) = 0.01, \quad \mathbb{P}(T \mid D) = 0.95, \quad \mathbb{P}(T^c \mid D^c) = 0.95.$$

Interpretation:

- The disease prevalence is 1%;
- The test correctly detects the disease 95% of the time;
- The test correctly returns a negative result 95% of the time for healthy individuals.

Question: A patient tests positive. What is $\mathbb{P}(D \mid T)$?
Using Bayes' Rule and the LOTP,

$$\mathbb{P}(D \mid T) = \frac{\mathbb{P}(T \mid D)\mathbb{P}(D)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T \mid D)\mathbb{P}(D)}{\mathbb{P}(T \mid D)\mathbb{P}(D) + \mathbb{P}(T \mid D^c)\mathbb{P}(D^c)} = \frac{0.95(0.01)}{0.95(0.01) + 0.05(0.99)} \approx 0.16.$$

Why This Feels Surprising

Most people expect the answer to be close to 95%.

But the disease is very rare.

Among many healthy people:

$$5\%$$

still test positive.

Those false positives greatly outnumber the true positives.

Lesson

Posterior probabilities depend both on:

- the evidence,
- and the prior probability.

Conditional Probability as Probability

Conditional Probability is a Probability Measure

Fix an event E with

$$\mathbb{P}(E) > 0.$$

Then the conditional probability function

$$\mathbb{P}(\cdot \mid E)$$

satisfies the three probability axioms.

Axiom 1 (Nonnegativity)

$$\mathbb{P}(A \mid E) \geq 0 \qquad \text{for every event } A.$$

Axiom 2 (Normalization)

$$\mathbb{P}(\mathcal{S} \mid E) = 1.$$

Axiom 3 (Countable Additivity) If $A_1, A_2, \ldots$ are pairwise disjoint events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \,\middle|\, E\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i \mid E).$$

Independence

Definition of Independence

Definition (Independence)

Events A and B are independent if

$$\mathbb{P}(A \mid B) = \mathbb{P}(A),$$

provided $\mathbb{P}(B) > 0$.

Interpretation:

Learning that B occurred gives no information about A (and vice versa*).

($\star$) We will later show that

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \quad \Longleftrightarrow \quad \mathbb{P}(B \mid A) = \mathbb{P}(B).$$

Thus independence is symmetric:

$$A \text{ is independent of } B \iff B \text{ is independent of } A.$$

Think of it this way: $\mathbb{P}(A \text{ with new (additional) information}) = \mathbb{P}(A \text{ with no additional information})$

Lemma (Equivalent characterization of independence)

Events A and B are independent if and only if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B).$$

Conditional Independence

Conditional Independence

Events A and B are conditionally independent given E if

$$\boxed{\mathbb{P}(A \cap B|E) = \mathbb{P}(A|E)\mathbb{P}(B|E).}$$

Conditional independence is different from ordinary independence.

Two events may:

- be independent but not conditionally independent;
- be conditionally independent but not independent.

Example: A Hidden Variable Creates Dependence

A coin is selected at random:

- Fair coin with probability $1/2$;
- Biased coin ($P(H) = 0.9$) with probability $1/2$.

If we know which coin was selected:
The tosses are independent.

If we do not know which coin was selected:
Observing early tosses changes our belief about which coin we have.
Therefore, earlier tosses affect our prediction of later tosses, so the tosses are *not* independent.

$$P(H_2 \mid H_1) \neq P(H_2).$$

Conditioning (Example)

Monty Hall Problem: Setup

Image source: Brilliant: Monty Hall Problem.

There are three doors:

- one door hides a car;
- two doors hide goats.

You choose one door, but it is not opened yet.

Assume, without loss of generality, that you initially choose door 1.

$$C_i = \{\text{car is behind door } i\}.$$

Before Monty opens a door,

$$\mathbb{P}(C_1) = \mathbb{P}(C_2) = \mathbb{P}(C_3) = \frac{1}{3}.$$

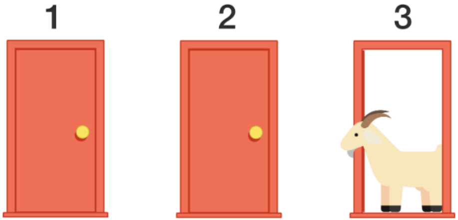


Image source: Brilliant: Monty Hall Problem.

Monty knows where the car is and always opens a door with a goat.

So the important question is:

Should you switch?

Car location	First choice	Switching
C_1	correct	loses
C_2	wrong	wins
C_3	wrong	wins

Therefore,

$$\mathbb{P}(\text{win by switching}) = \mathbb{P}(C_2 \cup C_3) = \mathbb{P}(C_2) + \mathbb{P}(C_3) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Monty Hall Problem via Bayes' Rule

Suppose Monty opens door 3.
Define

$$M_3 = \{\text{Monty opens door 3}\}.$$

Thus:

$$\underbrace{\mathbb{P}(\text{initial choice correct})}_{\text{door 1}} = \frac{1}{3}, \quad \text{and} \quad \underbrace{\mathbb{P}(\text{initial choice wrong})}_{\text{door 2 or 3}} = \frac{2}{3}.$$

After Monty reveals a goat behind door 3, the entire remaining probability mass concentrates on door 2.
Using Bayes' Rule,

$$\mathbb{P}(\text{win by switching}|M_3) = \mathbb{P}(C_2 \mid M_3) = \frac{\mathbb{P}(M_3 \mid C_2)\mathbb{P}(C_2)}{\mathbb{P}(M_3)}.$$

Now, $\mathbb{P}(C_2) = \frac{1}{3}$, $\mathbb{P}(M_3 \mid C_2) = 1$, and

$$\mathbb{P}(M_3) = \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{2}.$$

Therefore,

$$\mathbb{P}(C_2 \mid M_3) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}, \text{ and thus } \text{, and thus } \mathbb{P}(\text{win without switching}|M_3) = \mathbb{P}(C_1|M_3) = \frac{1}{3}$$

Summary

- Conditional probability updates probabilities using observed evidence.
- $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$
- Bayes' Rule relates:
$$\mathbb{P}(A|B) \quad \text{and} \quad \mathbb{P}(B|A).$$
- LOTP breaks problems into cases.
- Conditional probabilities themselves satisfy the probability axioms.
- Independence means:
$$\mathbb{P}(A|B) = \mathbb{P}(A).$$
- Conditioning is a powerful problem-solving strategy.

Next lecture

Random variables and distributions.