Chapter 4: Expectation

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- define expectation for discrete random variables;
- compute expectations from PMFs;
- use linearity of expectation;
- use indicator random variables and the fundamental bridge;
- define and compute variance and standard deviation;
- understand the Poisson approximation to the Binomial;
- know the expectations of the main discrete distributions.

Definition of Expectation

Why Expectation?

A distribution gives detailed information about a random variable:

$$\mathbb{P}(X = x), \qquad \mathbb{P}(X \le x), \qquad \mathbb{P}(a \le X \le b).$$

But sometimes we want a single number summarizing the "average" value of X.

Main idea

The expected value of a random variable is its probability-weighted average.

Idea: (Weighted) Averages

For ordinary numbers

$$x_1, x_2, \ldots, x_n,$$

the usual average is

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_j \tag{1}$$

More generally, if

$$p_1, \ldots, p_n \geq 0, \qquad \sum_{j=1}^{n} p_j = 1,$$

then the weighted average is

$$\sum_{j=1}^{n} x_j p_j.$$

Note that equation (1) uses equal weights $p_j = \frac{1}{n}$, $\forall j = 1, \ldots, n$, so it is a special case.

Definition: Expected Value

Definition (Expected value of a discrete r.v.)

Let X be a discrete random variable with possible values

$$x_1, x_2, \ldots .$$

The expected value of X is

$$\boxed{\mathbb{E}(X) = \sum_j x_j \mathbb{P}(X = x_j).}$$

Equivalently,

$$\mathbb{E}(X) = \sum_{x_j} x_j \, p_X(x_j),$$

where

$$p_X(x) = \mathbb{P}(X = x).$$

Interpretation

Expectation is the center of mass of the PMF (next slide).

Expectation as a Center of Mass (Proof is Optional)

Suppose a discrete random variable X takes values

$$x_1, x_2, \ldots$$

with probabilities

$$p_1, p_2, \ldots$$

Think of the PMF as placing masses p_i at locations x_i on the number line.
The **center of mass** (balance point) is a point c such that

$$\sum_i (x_i - c)p_i = 0.$$

This means:

- points to the right of c pull one way,
- points to the left of c pull the other way,
- and the system balances perfectly at c.

Expanding the equation:

$$\sum_i x_i p_i - c \sum_i p_i = 0.$$

Since

$$\sum_i p_i = 1,$$

we obtain

Example: One Fair Die

Let

$$X = \text{result of one fair die roll.}$$

Then

$$\mathbb{P}(X = k) = \frac{1}{6}, \qquad k = 1, 2, \ldots, 6.$$

Thus

$$\mathbb{E}(X) = \sum_{k=1}^{6} k\mathbb{P}(X = k) = \sum_{k=1}^{6} k \times \frac{1}{6} = \frac{1+2+3+4+5+6}{6}.$$

So

$$\mathbb{E}(X) = 3.5.$$

Important

The expectation does not have to be a possible value of X.

Example: Bernoulli

Let

$$X \sim \mathrm{Bern}(p).$$

Then

$$\mathbb{P}(X = 1) = p, \qquad \mathbb{P}(X = 0) = 1 - p.$$

Therefore

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

So

$$\mathbb{E}(X) = p.$$

Let

$$X \sim \mathrm{DUnif}(\{a, a+1, \ldots, b\}).$$

All values are equally likely.
There are

$$b - a + 1$$

possible values.
Thus

$$\mathbb{E}(X) = \frac{a + (a+1) + \cdots + b}{b - a + 1}.$$

The average of equally spaced endpoints is the midpoint:

$$\boxed{\mathbb{E}(X) = \frac{a+b}{2}.}$$

Example: Geometric Distribution

Let

$$X \sim \mathrm{Geom}(p).$$

Recall:

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

Then

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p.$$

Using the geometric-series identity,

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2}, \qquad |r| < 1,$$

with $r = 1 - p$, we obtain

$$\mathbb{E}(X) = \frac{1}{p}.$$

Example: Negative Binomial Distribution

Let

$$X \sim \mathrm{NegBin}(r, p),$$

where X counts the number of trials needed to obtain r successes.
Write

$$X = X_1 + \cdots + X_r,$$

where each

$$X_i \sim \mathrm{Geom}(p).$$

Therefore,

$$\mathbb{E}(X) = r \cdot \frac{1}{p}.$$

Hence

$$\mathbb{E}(X) = \frac{r}{p}.$$

Example: Poisson Distribution

Let

$$X \sim \text{Pois}(\lambda).$$

Recall:

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

Then

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!}.$$

Rearranging,

$$= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}.$$

Since

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = e^{\lambda},$$

we obtain

$$\mathbb{E}(X) = \lambda.$$

Proposition

If X and Y have the same distribution, then

$$\mathbb{E}(X) = \mathbb{E}(Y),$$

provided the expectations exist.

Reason: the formula

$$\mathbb{E}(X) = \sum_{x} x\mathbb{P}(X = x)$$

depends only on the PMF of X.

Warning

The converse is false: two random variables can have the same expectation but different distributions.

Linearity of Expectation

Linearity of Expectation

Theorem

For random variables X, Y and constant c,

$$\boxed{\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).}$$

Also,

$$\boxed{\mathbb{E}(cX) = c\mathbb{E}(X).}$$

More generally,

$$\mathbb{E}\left(\sum_{j=1}^{n} c_j X_j\right) = \sum_{j=1}^{n} c_j \mathbb{E}(X_j).$$

Application of Linearity in finding the Expectation of Bin(n,p)

Let

$$X \sim \text{Bin}(n, p).$$

Think of X as the number of successes in n Bernoulli trials.
Write

$$X = I_1 + \cdots + I_n,$$

where

$$I_j = \begin{cases} 1, & \text{success on trial } j, \\ 0, & \text{otherwise.} \end{cases}$$

Each

$$I_j \sim \text{Bern}(p),$$

so

$$\mathbb{E}(I_j) = p.$$

Therefore, by linearity of expectation,

$$\mathbb{E}(X) = \mathbb{E}[I_1 + \cdots + I_n] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_n] = \underbrace{p + \cdots + p}_{n \text{ times}} = np.$$

Note that we did not even use the PMF of $Ber(n, p)$ to find its expectation using the main formula, which made it much easier computationally.

Example: Hypergeometric Expectation

Let

$$X \sim \text{HGeom}(w, b, n),$$

where X is the number of white balls in a sample of size n drawn without replacement.
Write

$$X = I_1 + \cdots + I_n,$$

where $I_j = 1$ if the jth draw is white.
For each j,

$$\mathbb{P}(I_j = 1) = \frac{w}{w + b}.$$

So

$$\mathbb{E}(I_j) = \frac{w}{w + b}.$$

By linearity,

$$\boxed{\mathbb{E}(X) = n\frac{w}{w + b}.}$$

Key point

The draws are dependent, but linearity still works.

Monotonicity of Expectation

Theorem

If
$$X \geq Y$$
with probability 1, then
$$\mathbb{E}(X) \geq \mathbb{E}(Y).$$

Sketch of proof:
$$X - Y \geq 0.$$
Therefore,
$$\mathbb{E}(X - Y) \geq 0.$$
By linearity,
$$\mathbb{E}(X) - \mathbb{E}(Y) \geq 0.$$
For a concise proof, we need more technical tools that are not covered in this course.

Indicator Random Variables and the Fundamental Bridge

Indicator Random Variables

For an event A, define

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{if } A \text{ does not occur.} \end{cases}$$

Then

$$I_A \sim \text{Bern}(\mathbb{P}(A)).$$

Interpretation

Indicators convert events into random variables.

Basic Indicator Properties

For events A and B:

$$I_{A^c} = 1 - I_A.$$

$$I_{A \cap B} = I_A I_B.$$

$$I_{A \cup B} = I_A + I_B - I_A I_B.$$

Reason

Indicators behave like yes/no algebra:

$$1 = \text{yes}, \qquad 0 = \text{no}.$$

The Fundamental Bridge

Fundamental Bridge

For any event $A \subseteq \mathcal{S}$,

$$\boxed{\mathbb{E}(I_A) = \mathbb{P}(A).}$$

Proof:

$$I_A \sim \text{Bern}(\mathbb{P}(A)).$$

Since the expectation of a Bernoulli random variable is its success probability,

$$\mathbb{E}(I_A) = \mathbb{P}(A).$$

Meaning

Probability questions can often be turned into expectation questions, and vice versa.

Example: Matching Problem

A deck has cards labeled

$$1, 2, \ldots, n.$$

After shuffling, a match occurs if card j is in position j.
Let

$$X = \text{number of matches}.$$

Define

$$I_j = \begin{cases} 1, & \text{card } j \text{ is in position } j, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$X = I_1 + \cdots + I_n.$$

Since

$$\mathbb{P}(I_j = 1) = \frac{1}{n},$$

we get

$$\mathbb{E}(X) = n \cdot \frac{1}{n} = 1.$$

$$\boxed{\mathbb{E}(X) = 1.}$$

Example: Birthday Matches

Suppose there are n people.
Let

$$X = \text{number of pairs of people with the same birthday.}$$

There are

$$\binom{n}{2}$$

pairs.
For each pair (i, j), define

$$I_{ij} = \begin{cases} 1, & \text{people } i, j \text{ have the same birthday,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$X = \sum_{i<j} I_{ij}.$$

Since

$$\mathbb{E}(I_{ij}) = \mathbb{P}(I_{ij} = 1) = \frac{1}{365},$$

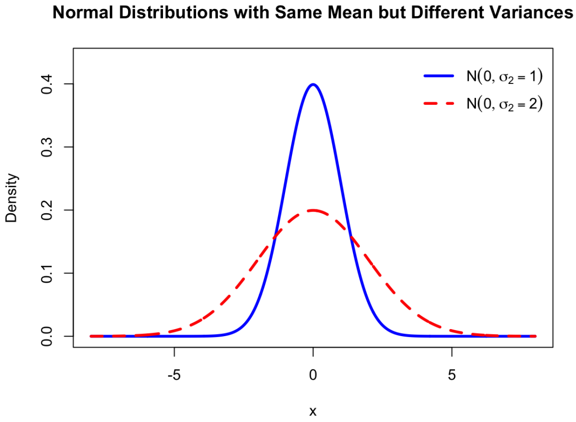
we have

$$\mathbb{E}(X) = \binom{n}{2} \frac{1}{365}.$$

Variance

Why Variance?

Expectation measures the center of a distribution.
But two distributions can have the same expectation and very different spread.
We need a numerical measure of variability.
Variance measures the average squared distance from the mean.



Definition: Variance and Standard Deviation

Definition

The variance of X is

$$\boxed{\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}X)^2\right].}$$

Variance is a measure of dispersion (from the mean). It is a measure of how far a set of numbers are spread out from their mean, on average.

The standard deviation is

$$\boxed{\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}.}$$

Variance has squared units; standard deviation returns to the original units.

Motivation behind the formula $\mathbb{E}[X - \mathbb{E}[X]]^2$?

Consider the function

$$g(c) = \mathbb{E}\big[(X - c)^2\big].$$

The quantity $g(c)$ measures the average squared distance between X and the constant c.
We claim that $g(c)$ is minimized when

$$c = \mathbb{E}[X].$$

Expand the square:

$$\begin{aligned} g(c) &= \mathbb{E}\left[(X - \mathbb{E}[X] + \mathbb{E}[X] - c)^2\right] \\ &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] + 2(\mathbb{E}[X] - c)\,\mathbb{E}[X - \mathbb{E}[X]] \\ &\quad + (\mathbb{E}[X] - c)^2. \end{aligned}$$

Since $\mathbb{E}[X - \mathbb{E}[X]] = 0$,, we obtain $g(c) = \mathrm{Var}(X) + (\mathbb{E}[X] - c)^2$.
Because

$$(\mathbb{E}[X] - c)^2 \geq 0,$$

the minimum occurs when

$$c = \mathbb{E}[X].$$

Thus, the expectation is the *best constant predictor* of X in mean squared error.

Alternative Formula for Variance

Variance Formula

$$\boxed{\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2.}$$

Proof : Let

$$\mu = \mathbb{E}X.$$

Then

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right].$$

Expanding:

$$\mathbb{E}(X^2 - 2\mu X + \mu^2) = \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2.$$

Since $\mu = \mathbb{E}X$,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2.$$

Properties of Variance

Shifting does not change spread. Scaling by b scales squared distances by b^2.

$$\boxed{\mathrm{Var}(X+a) = \mathrm{Var}(X) \quad \textit{and} \quad \mathrm{Var}(bX) = b^2\,\mathrm{Var}(X).}$$

Warning

Unlike Expectation, Variance is not linear, i.e., in general,

$$\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y).$$

In fact,

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \tag{2}$$

and for $n > 1$ random variables,

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \tag{3}$$

We will study this in further depth in Chapter 7 (Joint Distributions).

Variance of a Binomial Random Variable

Let

$$X \sim \text{Bin}(n, p).$$

Write

$$X = I_1 + \cdots + I_n,$$

where the I_j are independent Bernoulli(p) random variables.
Each has variance

$$\text{Var}(I_j) = p(1-p).$$

Since the indicators are independent (*),

$$\text{Var}(X) = \sum_{j=1}^{n} \text{Var}(I_j).$$

Therefore

$$\text{Var}(X) = np(1-p).$$

(*): Note that we previously proved that the expected value of a sum of random variables is equal to the sum of their expected values. However, this property does *not* generally hold for variance. In general, the variance of a sum is *not* equal to the sum of the variances. Variances add only in the special case where the random variables have zero covariance, that is, when they are uncorrelated. In particular, independence implies zero covariance, so variances of independent random variables do add. We will study covariance, correlation, and independence in more detail later. For a direct derivation of the variance of the Binomial distribution using its PMF, click here.

A Useful Variance Formula (exercise)

Show that for any random variable X,

$$\mathsf{Var}(X) = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2.$$

Discrete Uniform Variance

Let

$$X \sim \mathrm{DUnif}\{1, 2, \ldots, n\}.$$

Then

$$\mathbb{E}[X] = \frac{n+1}{2}.$$

Also,

$$\mathbb{E}[X^2] = \frac{1}{n}\sum_{k=1}^{n} k^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}.$$

Therefore,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

$$\mathrm{Var}(X) = \frac{n^2-1}{12}.$$

Poisson Variance

Let

$$X \sim \mathrm{Pois}(\lambda).$$

Then

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

We use

$$\mathbb{E}[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)e^{-\lambda}\frac{\lambda^k}{k!}.$$

For $k \geq 2$,

$$k(k-1)\frac{\lambda^k}{k!} = \frac{\lambda^k}{(k-2)!}.$$

Thus,

$$\mathbb{E}[X(X-1)] = \lambda^2 e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda^2.$$

Poisson Variance Continued

Since

$$\mathbb{E}[X] = \lambda,$$

we get

$$\mathrm{Var}(X) = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2.$$

Therefore,

$$\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2.$$

$$\mathrm{Var}(X) = \lambda.$$

So for a Poisson random variable,

$$\mathbb{E}[X] = \mathrm{Var}(X) = \lambda.$$

Geometric Variance

Let

$$X \sim \mathrm{Geom}(p),$$

where

$$X = \text{number of trials until the first success.}$$

Thus,

$$\mathbb{P}(X = k) = (1-p)^{k-1} p, \qquad k = 1, 2, 3, \ldots$$

Let $q = 1 - p$. Then

$$\mathbb{E}[X] = \frac{1}{p}.$$

A standard power series identity gives

$$\sum_{k=1}^{\infty} k^2 q^{k-1} = \frac{1+q}{(1-q)^3}.$$

Therefore,

$$\mathbb{E}[X^2] = p \sum_{k=1}^{\infty} k^2 q^{k-1} = p \cdot \frac{1+q}{(1-q)^3}.$$

Geometric Variance Continued

Since

$$1 - q = p,$$

we have

$$\mathbb{E}[X^2] = p \cdot \frac{1+q}{p^3} = \frac{1+q}{p^2}.$$

Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

So

$$\mathrm{Var}(X) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Since $q = 1 - p$,

$$\mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Negative Binomial Variance

Let

$$X \sim \text{NegBin}(r, p),$$

where

$$X = \text{number of trials needed to obtain } r \text{ successes.}$$

A Negative Binomial random variable can be written as a sum of r independent Geometric random variables:

$$X = G_1 + G_2 + \cdots + G_r,$$

where each

$$G_i \sim \text{Geom}(p).$$

Each G_i is the waiting time for one additional success.

Negative Binomial Variance Continued

Since

$$X = G_1 + \cdots + G_r$$

and the G_i are independent,

$$\mathrm{Var}(X) = \mathrm{Var}(G_1) + \cdots + \mathrm{Var}(G_r).$$

From the Geometric variance formula,

$$\mathrm{Var}(G_i) = \frac{1-p}{p^2}.$$

Therefore,

$$\mathrm{Var}(X) = r \cdot \frac{1-p}{p^2}.$$

$$\mathrm{Var}(X) = \frac{r(1-p)}{p^2}.$$

Hypergeometric Variance

Let
$$X \sim \text{HGeom}(w, b, n),$$
where an urn has w white balls and b black balls.
Let
$$N = w + b.$$
We draw n balls without replacement, and
$$X = \text{number of white balls selected.}$$
Write
$$X = I_1 + I_2 + \cdots + I_n,$$
where
$$I_j = \begin{cases} 1, & \text{if draw } j \text{ is white,} \\ 0, & \text{otherwise.} \end{cases}$$

Hypergeometric Variance Continued

Each indicator satisfies

$$\mathbb{E}[I_j] = \frac{w}{N}.$$

This feels straightforward (naive probability), but I think we should not be that comfortable with it because the draws are dependent (without replacement)! So the j^{th} draw depends on the $(j-1)^{th}$ draw. But you can prove (by conditioning, on the first draw) that the second draw has expected value w/N, and so on for the next draw, by conditioning on the previous draws.
So

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[I_j] = \sum_{i=1}^{n} w/N = n\frac{w}{N}.$$

Also,

$$\mathrm{Var}(I_j) = \frac{w}{N}\left(1 - \frac{w}{N}\right).$$

Because sampling is without replacement, the indicators are not independent.
For $i \neq j$,

$$\mathrm{Cov}(I_i, I_j) = -\frac{w}{N}\frac{b}{N}\frac{1}{N-1}.$$

Hypergeometric Variance Formula

Using

$$\mathrm{Var}(X) = \sum_{j=1}^{n} \mathrm{Var}(I_j) + 2\sum_{i<j} \mathrm{Cov}(I_i, I_j),$$

we get

$$\mathrm{Var}(X) = n\frac{w}{N}\frac{b}{N} - n(n-1)\frac{w}{N}\frac{b}{N}\frac{1}{N-1}.$$

Factor:

$$\mathrm{Var}(X) = n\frac{w}{N}\frac{b}{N}\left(1 - \frac{n-1}{N-1}\right).$$

Therefore,

$$\mathrm{Var}(X) = n\frac{w}{N}\frac{b}{N}\frac{N-n}{N-1}.$$

Nonnegativity of Variance

Proposition

For any r.v. X,
$$\mathrm{Var}(X) \geq 0.$$

Reason:
$$(X - \mathbb{E}X)^2 \geq 0.$$

Thus its expectation is nonnegative.
Later, we will learn the Jensen's Inequality, which implies this Proposition as a special case.
Also,
$$\mathrm{Var}(X) = 0$$
if and only if X is constant with probability 1.

Poisson Approximation to Binomial

Poisson Approximation to Binomial

When

$$X \sim \text{Bin}(n, p).$$

AND

- n is large;
- p is small;
- $\lambda = np$ is moderate,

Then,

$$\boxed{X \approx \text{Pois}(\lambda).}$$

That is,

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda} \frac{\lambda^k}{k!}.$$

Why the Poisson Approximation Works

Assume $X \sim \text{Bin}(n, p)$ with $p = \lambda/n$, where $\lambda > 0$ is fixed. For each fixed $k = 0, 1, 2, \ldots,$ we have

$$\mathbb{P}_{Binom(n,p)}(X = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{n!}{k!(n-k)!}\frac{\lambda^k}{n^k}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-k}$$

$$= \underbrace{\left[(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]}_{\longrightarrow 1} \underbrace{\frac{\lambda^k}{k!}}_{\text{stays fixed}} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\longrightarrow e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\longrightarrow 1}$$

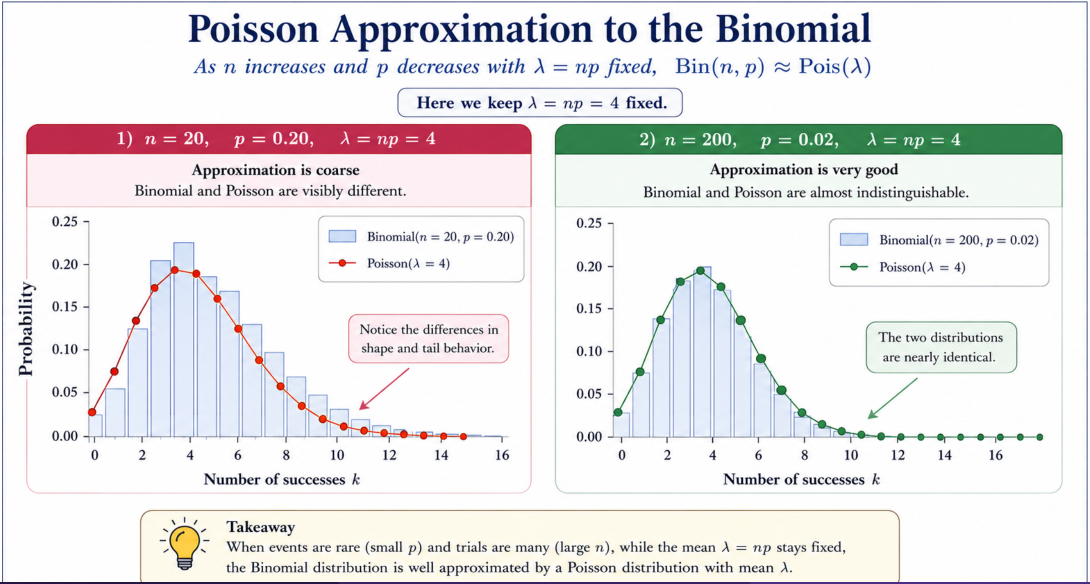
$$\longrightarrow 1 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot 1 = e^{-\lambda}\frac{\lambda^k}{k!} = \mathbb{P}_{X\sim Poisson(\lambda)}(X = k)$$

Thus,

$$\mathbb{P}_{X\sim Bin(n,p)}(X = k) \xrightarrow{n\to\infty} e^{-\lambda}\frac{\lambda^k}{k!}.$$

This phenomenon is formally known as convergence in distribution (or weak convergence) in Probability theory. We will see more with Central Limit Theorem (CLT) and Law of Large Numbers (LLN) in Chapter 10B (Limit Theorems).

Visualization of Poisson Approximation to the Binomial



Example: Website Visitors

Suppose 1,000,000 people independently decide whether to visit a website.
Each visits with probability

$$p = 2 \times 10^{-6}.$$

Let

$$X = \text{number of visitors}.$$

Then

$$X \sim \text{Bin}(1{,}000{,}000, 2 \times 10^{-6}).$$

Here

$$\lambda = np = 2.$$

So

$$X \approx \text{Pois}(2).$$

For example,

$$\mathbb{P}(X = 0) \approx e^{-2}.$$

Note that it is computationally hard to compute $\mathbb{P}[X = k]$ directly from the Binomial PMF for different values of $k \in \mathbb{N}$:

$$\mathbb{P}(X = k) = \frac{1000000!}{k!(1000000 - k)!}\left(\frac{2}{10^6}\right)^k\left(1 - \frac{2}{10^6}\right)^{1000000-k} \tag{4}$$

Summary

- Expectation is a probability-weighted average:

$$\mathbb{E}(X) = \sum_x x\mathbb{P}(X = x).$$

- Linearity of Expectation:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y),$$

  with no independence required. Also works for linear combinations of finite number of random variabels.
- Fundamental bridge:

$$\mathbb{E}(I_A) = \mathbb{P}(A).$$

- Variance measures spread:

$$\begin{aligned}\mathrm{Var}(X) &\triangleq \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - \mu^2 \\ &= \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2 = \mathbb{E}[X(X-1)] + \mu - \mu^2.\end{aligned}$$

  with no linearity property, except under zero correlation (independence).
- Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is moderate.

Next lecture

Continuous random variables