Chapter 5A: Continuous Random Variables, PDFs, and Uniform Distribution

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- define and interpret PDFs and CDFs;
- compute probabilities by integrating a PDF;
- recognize valid PDFs;
- compute expectation for continuous random variables;
- define and use the Uniform distribution;
- understand the universality of the Uniform distribution.

Continuous Random Variables

From Discrete to Continuous Random Variables

Discrete random variables take values in a finite or countably infinite set.

Continuous random variables can take any value in an interval, such as

$$(0, 1), \qquad (a, b), \qquad [0, \infty), \qquad \mathbb{R}.$$

Main difference

For a continuous random variable,

$$\mathbb{P}(X = x) = 0$$

for every individual point x.

Probability Density Functions

Definition

For a continuous random variable X, the probability density function, or PDF, is a function f such that

$$\mathbb{P}(X \in A) = \int_A f(x)\, dx, \quad \forall A \subseteq \mathbb{R}$$

The value $f(x)$ itself is not a probability (could be greater than 1).

Probabilities are areas under the density curve.

How to verify a function is a PDF?

A function f is a valid PDF if it satisfies two properties, namely

$$f(x) \geq 0 \qquad \text{for all } x,$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Analogy

For discrete random variables:

$$p_X(x) \geq 0, \quad \sum_x p_X(x) = 1.$$

For continuous random variables:

$$f_X(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

CDF and PDF

CDF has the same formula for continuous and discrete r.v.s, i.e.

$$F(x) = \mathbb{P}(X \leq x).$$

For a continuous random variable with PDF f, according to the definition of PDF,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Note that , according to the definition of CDF,

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

This, together with the Fundamental Theorem of Calculus, implies that

$$\frac{d}{dx}F(x) = F'(x) = f(x) \tag{1}$$

For **continuous** r.v.s, since $\mathbb{P}(X = a) = 0$ for any $a \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}(X \leq a) &= \mathbb{P}(X < a), \\
\mathbb{P}(X \geq a) &= \mathbb{P}(X > a), \\
\mathbb{P}(a \leq X \leq b) &= \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X < b).
\end{aligned}$$

What Does the Density $f(x)$ Mean?

For a continuous random variable,

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\,dt.$$

For a small number $\epsilon > 0$,

$$\mathbb{P}(x \leq X \leq x + \epsilon) = F(x + \epsilon) - F(x).$$

Therefore,

$$\frac{\mathbb{P}(x \leq X \leq x + \epsilon)}{\epsilon} = \frac{F(x + \epsilon) - F(x)}{\epsilon}.$$

Taking the limit as $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \frac{\mathbb{P}(x \leq X \leq x + \epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{F(x + \epsilon) - F(x)}{\epsilon}.$$

By the definition of derivative,

$$f(x) = F'(x) = \lim_{\epsilon \to 0} \frac{F(x + \epsilon) - F(x)}{\epsilon}.$$

Thus,

$$f(x) = \lim_{\epsilon \to 0} \frac{\mathbb{P}(x \leq X \leq x + \epsilon)}{\epsilon}$$

Expectation for Continuous Random Variables

Definition

If X is continuous with PDF f, then

$$\boxed{\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx.}$$

If the support is an interval, we integrate over that interval.

Expectation is still the balancing point of the distribution.

Continuous LOTUS

LOTUS

If X has PDF f and $g : \mathbb{R} \to \mathbb{R}$, then

$$\boxed{\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx.}$$

Proof idea. For a small interval $[x, x + \Delta x]$,

$$\mathbb{P}(x \leq X \leq x + \Delta x) \approx f(x)\Delta x.$$

On this interval, $g(X)$ is approximately $g(x)$. Hence,

$$g(X)\mathbb{P}(x \leq X \leq x + \Delta x) \approx g(x)f(x)\Delta x.$$

Adding over many small intervals,

$$\mathbb{E}(g(X)) \approx \sum_i g(x_i)f(x_i)\Delta x.$$

Taking the limit as $\Delta x \to 0$, the Riemann sum becomes

$$\mathbb{E}(g(X)) = \int^{\infty} g(x)f(x)\,dx.$$

Uniform Distribution

Uniform Distribution

Definition

A random variable U has the Uniform distribution on (a, b) if its PDF is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

We write

$$U \sim \text{Unif}(a, b).$$

PDF Verification

$$f(x) = \frac{1}{b-a} > 0$$

since $b > a$. And

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(b-a) = 1.$$

Uniform CDF

If

$$U \sim \text{Unif}(a, b),$$

then its CDF is

$$F(x) = \begin{cases} 0, & x \le a, \\ \dfrac{x-a}{b-a}, & a < x < b, \\ 1, & x \ge b. \end{cases}$$

For the standard Uniform,

$$U \sim \text{Unif}(0, 1),$$

we have

$$F(x) = x, \qquad 0 < x < 1.$$

Universality of the Uniform

Why the Uniform Distribution Is Special

The Uniform distribution is a universal starting point for simulation.

Many algorithms first generate

$$U \sim \mathrm{Unif}(0, 1),$$

then transform U to obtain another distribution.

Main idea

A Uniform random variable can be transformed into a random variable with many other distributions.

Inverse CDF Method

Suppose F is a continuous, strictly increasing CDF.
Let

$$U \sim \text{Unif}(0, 1).$$

Define

$$X = F^{-1}(U).$$

Then

$$X \text{ has CDF } F.$$

This is called the inverse CDF method or inverse transform sampling.

Why the Inverse CDF Method Works

Let

$$X = F^{-1}(U).$$

Then

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x).$$

Since F is increasing,

$$F^{-1}(U) \leq x \iff U \leq F(x).$$

Therefore,

$$\mathbb{P}(X \leq x) = \mathbb{P}(U \leq F(x)).$$

Because $U \sim \text{Unif}(0, 1)$,

$$\mathbb{P}(U \leq F(x)) = F(x).$$

Thus X has CDF F.

Conversely, if X has continuous strictly increasing CDF F, then

$$F(X) \sim \mathrm{Unif}(0, 1).$$

Interpretation

Applying the CDF to a random variable converts its value into a percentile.

For example, if exam scores have CDF F, then $F(X)$ is the percentile of a randomly selected student.

Uniform Mean and Variance

If $U \sim \text{Unif}(a, b)$, then

$$f_U(u) = \frac{1}{b-a}, \qquad a \le u \le b.$$

Mean

$$\mathbb{E}(U) = \int_a^b u \, \frac{1}{b-a} \, du = \frac{1}{b-a} \left[\frac{u^2}{2} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

Variance

First,

$$\mathbb{E}(U^2) = \int_a^b u^2 \, \frac{1}{b-a} \, du = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}.$$

Thus,

$$\text{Var}(U) = \mathbb{E}(U^2) - [\mathbb{E}(U)]^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2} \right)^2 = \frac{(b-a)^2}{12}.$$

Location-Scale Transformation

If

$$U \sim \text{Unif}(0, 1),$$

then

$$a + (b - a)U \sim \text{Unif}(a, b).$$

This transforms:

$$(0, 1) \quad \longrightarrow \quad (a, b).$$

Useful idea

Start with the standard Uniform and shift/scale to get any Uniform interval.

Summary

- Continuous random variables have probabilities over intervals, not points.
- A PDF integrates to probabilities:
$$\mathbb{P}(a < X < b) = \int_a^b f(x)\,dx.$$
- A valid PDF is nonnegative and integrates to 1.
- The CDF is accumulated area:
$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$
- Uniform probabilities are proportional to interval length.
- The Uniform distribution is universal through the inverse CDF method.

Next lecture

Normal and Exponential distributions.