Chapter 8A: Transformations and Convolutions

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- compute distributions of transformed random variables;
- apply the change-of-variables formula;
- distinguish between discrete and continuous transformations;
- derive distributions using the CDF method;
- define convolutions;
- compute distributions of sums of independent random variables;
- derive Gamma and Triangle distributions from convolutions.

Transformations of Random Variables

Why Transform Random Variables?

Transformations appear everywhere in probability and statistics.

Examples:

- unit conversion;
- standardization;
- logarithms;
- sums and averages;
- squares and absolute values;
- maxima and minima.

Main question

If

$$Y = g(X),$$

what is the distribution of Y?

Transformations in the Discrete Case

If X is discrete and

$$Y = g(X),$$

then

$$\mathbb{P}(Y = y) = \sum_{x : g(x) = y} \mathbb{P}(X = x).$$

One-to-one case

If g is one-to-one, then

$$\mathbb{P}(Y = y) = \mathbb{P}(X = g^{-1}(y)).$$

Transformation in the Continuous Case

Setup

Let X be a continuous random variable with PDF f_X and support

$$S_X = \{x : f_X(x) > 0\}.$$

Let

$$Y = g(X),$$

where $g : S_X \to S_Y$ is differentiable and strictly monotone (i.e., strictly increasing or strictly decreasing). Then g has an inverse

$$g^{-1} : S_Y \to S_X.$$

We write

$$x = g^{-1}(y).$$

The support of Y is

$$S_Y = g(S_X) = \{g(x) : x \in S_X\}.$$

In the next slide, we will see how to find f_Y.

Change of Variable Formula

Change of Variables Formula

For $y \in S_Y$,

$$\boxed{f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right|}$$

and for $y \notin S_Y$,

$$f_Y(y) = 0.$$

Proof: g Increasing

Assume g is strictly increasing on S_X. Then

$$Y = g(X)$$

has support

$$S_Y = g(S_X),$$

and for $y \in S_Y$,

$$Y \leq y \iff g(X) \leq y \iff X \leq g^{-1}(y).$$

Therefore, since g is increasing,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}\big(X \leq g^{-1}(y)\big) = F_X\big(g^{-1}(y)\big).$$

Now differentiate with respect to y. According to the chain rule, we have

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\big(g^{-1}(y)\big).$$

Using the chain rule,

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \frac{d}{dy} g^{-1}(y).$$

Since g is increasing,

$$\frac{d}{dy} g^{-1}(y) > 0,$$

so

$$\boxed{f_Y(y) = f_X\big(g^{-1}(y)\big) \left| \frac{d}{dy} g^{-1}(y) \right|.}$$

Proof: g Decreasing

Assume g is strictly decreasing on S_X. Then for $y \in S_Y$,

$$Y \le y \iff g(X) \le y.$$

Because g is decreasing, applying g^{-1} reverses the inequality:

$$g(X) \le y \iff X \ge g^{-1}(y).$$

Therefore,

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}\big(X \ge g^{-1}(y)\big) = 1 - F_X\big(g^{-1}(y)\big).$$

Differentiate with respect to y,

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X\big(g^{-1}(y)\big)\right].$$

Using the chain rule,

$$f_Y(y) = -f_X\big(g^{-1}(y)\big)\frac{d}{dy}g^{-1}(y).$$

Since g is decreasing,

$$\frac{d}{dy}g^{-1}(y) < 0.$$

Hence,

$$-\frac{d}{dy}g^{-1}(y) = \left|\frac{d}{dy}g^{-1}(y)\right|.$$

Thus,

$$\boxed{f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{d}{dy}g^{-1}(y)\right|.}$$

Transformation Formula: Final Form

Let

$$Y = g(X),$$

where $g : S_X \to S_Y$ is differentiable and strictly monotone (i.e., strictly increasing or strictly decreasing).
Then for $y \in S_Y$,

$$\boxed{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|}$$

where

$$x = g^{-1}(y).$$

Equivalently,

$$\boxed{f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|.}$$

For $y \notin S_Y$,

$$f_Y(y) = 0.$$

Important

The absolute value appears because the inverse derivative is positive in the increasing case and negative in the decreasing case.

How to Use the Formula

Steps:

1. Write
$$y = g(x).$$
2. Solve for
$$x = g^{-1}(y).$$
3. Compute
$$\left|\frac{dx}{dy}\right|.$$
4. Substitute everything in terms of y.

Do not forget

Always specify the support of the transformed variable.

Log-Normal Distribution

Let
$$X \sim N(0,1),$$
and define
$$Y = e^X.$$
Here, $g(x) = e^x$ is strictly increasing, so it satisfies the condition of the Change of Variable Formula. Note that
$$x = \log y.$$
This means that
$$g^{-1}(y) = \log(y)$$
Also,
$$\frac{dx}{dy} = \frac{1}{y}.$$
Thus,
$$f_Y(y) = \varphi(\log y)\frac{1}{y}, \qquad y > 0.$$
This is called the Log-Normal distribution.

Squaring a Standard Normal

Let

$$X \sim N(0,1),$$

and define

$$Y = X^2.$$

The function

$$g(x) = x^2$$

is not one-to-one.

So the simple change-of-variables formula does not directly apply.

CDF Method for $Y = X^2$

For

$$y > 0,$$

the event

$$X^2 \leq y$$

is equivalent to

$$-\sqrt{y} \leq X \leq \sqrt{y}.$$

Thus,

$$F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}).$$

Using symmetry,

$$F_Y(y) = 2\Phi(\sqrt{y}) - 1.$$

Differentiating,

$$f_Y(y) = 2\varphi(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}.$$

Hence,

$$\boxed{f_Y(y) = \frac{\varphi(\sqrt{y})}{\sqrt{y}}, \qquad y > 0.}$$

This is an example of a Chi-Square distribution.

Location-Scale Transformations

Suppose

$$Y = a + bX, \qquad b \neq 0.$$

Then

$$x = \frac{y-a}{b},$$

and

$$\left|\frac{dx}{dy}\right| = \frac{1}{|b|}.$$

Thus,

$$f_Y(y) = f_X\left(\frac{y-a}{b}\right)\frac{1}{|b|}.$$

Convolutions

What Is a Convolution?

A convolution is the distribution of a sum of independent random variables.

If

$$T = X + Y,$$

and X, Y are independent, we want the distribution of T.

Why sums matter

Sums appear everywhere:

- totals;
- averages;
- waiting times;
- counting processes.

Discrete Convolution Formula

Theorem

If X and Y are independent discrete random variables, then

$$\boxed{\mathbb{P}(T = t) = \sum_x \mathbb{P}(Y = t - x)\mathbb{P}(X = x).}$$

This is just the law of total probability.

Continuous Convolution Formula

Theorem

If X and Y are independent continuous random variables, then

$$\boxed{f_T(t) = \int_{-\infty}^{\infty} f_Y(t-x) f_X(x)\, dx.}$$

This is the continuous analog of the discrete convolution sum.

Convolution of Two Exponentials

Let

$$X, Y \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda).$$

Define

$$T = X + Y.$$

Then for

$$t > 0,$$

$$f_T(t) = \int_0^t \lambda e^{-\lambda(t-x)} \lambda e^{-\lambda x} dx.$$

Gamma Distribution Appears

Simplifying,

$$f_T(t) = \lambda^2 e^{-\lambda t} \int_0^t dx.$$

Therefore,

$$\boxed{f_T(t) = \lambda^2 t e^{-\lambda t}, \qquad t > 0.}$$

This is a Gamma distribution.

Convolution of Two Uniforms

Let

$$X, Y \overset{\text{i.i.d.}}{\sim} \text{Unif}(0, 1).$$

Define

$$T = X + Y.$$

Then

$$0 < T < 2.$$

The PDF becomes piecewise linear.

Triangle Distribution

The PDF of

$$T = X + Y$$

is

$$f_T(t) = \begin{cases} t, & 0 < t \leq 1, \\ 2 - t, & 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Interpretation

Middle values are more likely because there are more ways to obtain them.

Why the Triangle Shape Makes Sense

Values near

$$1$$

can occur in many ways:

$$0.4 + 0.6, \qquad 0.2 + 0.8, \qquad 0.5 + 0.5,$$

etc.

But values near

$$0$$

or

$$2$$

require both variables to be extreme simultaneously.

This explains the triangular shape.

Connection with Dice

Recall: the sum of two fair dice also produced a triangular PMF.

Analogy

$$\text{Discrete Uniform} \quad \longrightarrow \quad \text{triangular PMF,}$$

$$\text{Continuous Uniform} \quad \longrightarrow \quad \text{triangular PDF.}$$

Summary

- Transformations convert one distribution into another.
- Continuous transformations require Jacobians.
- The CDF method always works.
- Convolutions describe sums of independent random variables.
- Sums of Exponentials lead to Gamma distributions.
- Sums of Uniforms lead to Triangle distributions.

Next lecture

Beta and Gamma distributions.