Chapter 8B: Beta and Gamma Distributions

MATH/STAT 394: Probability I

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Learning Goals

By the end of this lecture, students should be able to:

- define the Beta distribution;
- understand how Beta parameters affect shape;
- recognize Uniform distributions as special Beta distributions;
- define the Gamma function and Gamma distribution;
- interpret Gamma distributions as sums of Exponential random variables;
- compute means and variances of Beta and Gamma random variables;
- understand the relationship between Gamma and Poisson processes.

Beta Distribution

Why the Beta Distribution?

Many random quantities naturally live between

$$0 \quad \text{and} \quad 1.$$

Examples:

- probabilities;
- proportions;
- percentages;
- fractions;
- success rates.

Main idea

The Beta distribution is one of the most important distributions on the interval

$$(0, 1).$$

Beta Function

Definition

For

$$a, b > 0,$$

the Beta function is

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

The Beta function serves as a normalization constant.

Definition

A random variable X has the Beta distribution with parameters

$$a, b > 0$$

if its PDF is

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \qquad 0 < x < 1.$$

Notation:

$$X \sim \mathrm{Beta}(a,b).$$

Shape of Beta Distributions

Different parameter values create very different shapes.

Examples:

- $$a = b = 1$$
  gives a Uniform distribution;
- $$a, b > 1$$
  produces mound-shaped densities;
- $$a < 1$$
  or
  $$b < 1$$
  can create spikes near the endpoints.

Important

The Beta family is extremely flexible.

If

$$a = b = 1,$$

then

$$B(1,1) = 1,$$

and the PDF becomes

$$f(x) = 1, \qquad 0 < x < 1.$$

Thus:

$$\text{Beta}(1,1) = \text{Unif}(0,1).$$

Mean and Variance of Beta

If

$$X \sim \text{Beta}(a, b),$$

then

$$\mathbb{E}(X) = \frac{a}{a+b}.$$

Also,

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Interpretation

The parameters control both:

- the center;
- the concentration.

Beta Distribution and Beliefs

Suppose a coin has unknown success probability

$$p.$$

Before observing data, we may model uncertainty about p using a Beta distribution.

Example:

$$p \sim \text{Beta}(2, 2)$$

represents a prior belief favoring values near

$$0.5.$$

Bayesian interpretation

Beta distributions are natural models for uncertain probabilities.

Beta-Binomial Connection

Suppose:

- $$p \sim \mathrm{Beta}(a, b),$$

  and conditional on p,

- $$X \mid p \sim \mathrm{Bin}(n, p).$$

Then the posterior distribution of p given $X = k$ is again Beta:

$$\boxed{p \mid X = k \sim \mathrm{Beta}(a + k, b + n - k).}$$

This is called conjugacy.

Bayes' Billiards

Imagine tossing a ball uniformly onto a billiard table.
A second sequence of balls lands independently.

We ask: what fraction land to the left of the first ball?

This naturally leads to Beta distributions.

Historical note

This is one of the earliest Bayesian probability models.

Gamma Function

Recall:

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Can factorials be extended to non-integer values?

Answer

Yes. The Gamma function generalizes factorials to positive real numbers.

Definition

For

$$\alpha > 0,$$

the Gamma function is

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Connection with Factorials

Theorem

For positive integers n,

$$\boxed{\Gamma(n) = (n-1)!.}$$

Proof. Show $\Gamma(n) = (n-1) \times \Gamma(n-1)$ using integration by parts, and then solve the recursion with induction.
Examples:

$$\Gamma(1) = 1,$$

$$\Gamma(2) = 1!,$$

$$\Gamma(3) = 2!,$$

etc.

Gamma Distribution

Definition of Gamma Distribution

Definition

A random variable X has the Gamma distribution with parameters

$$\alpha, \lambda > 0$$

if its PDF is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0.$$

Notation:

$$X \sim \text{Gamma}(\alpha, \lambda).$$

Exponential Distribution as a Gamma Distribution

If

$$\alpha = 1,$$

then

$$\Gamma(1) = 1,$$

and the PDF becomes

$$f(x) = \lambda e^{-\lambda x}.$$

Thus:

$$\text{Gamma}(1, \lambda) = \text{Exp}(\lambda).$$

Mean and Variance of Gamma

If

$$X \sim \mathsf{Gamma}(\alpha, \lambda),$$

then

$$\mathbb{E}(X) = \frac{\alpha}{\lambda}.$$

Also,

$$\mathsf{Var}(X) = \frac{\alpha}{\lambda^2}.$$

Proof: Mean and Variance of Gamma

The density is

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0.$$

Then

$$\mathbb{E}(X) = \int_0^\infty x f_X(x)\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x}\, dx.$$

Using

$$\int_0^\infty x^\alpha e^{-\lambda x}\, dx = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}},$$

we get

$$\mathbb{E}(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}.$$

Since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$,

$$\mathbb{E}(X) = \frac{\alpha}{\lambda}.$$

Proof: Variance of Gamma

First compute

$$\mathbb{E}(X^2) = \int_0^\infty x^2 f_X(x)\,dx.$$

Thus

$$\mathbb{E}(X^2) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x}\,dx.$$

Using

$$\int_0^\infty x^{\alpha+1} e^{-\lambda x}\,dx = \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}},$$

we get

$$\mathbb{E}(X^2) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}}.$$

Since

$$\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha),$$

we have

$$\mathbb{E}(X^2) = \frac{\alpha(\alpha+1)}{\lambda^2}.$$

Therefore,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2}.$$

Hence

Gamma as a Sum of Exponentials

Suppose

$$X_1, \ldots, X_n$$

are independent Exponential(λ) random variables.
Define

$$T = X_1 + \cdots + X_n.$$

Then:

$$T \sim \text{Gamma}(n, \lambda).$$

Interpretation

Gamma distributions describe waiting times for multiple arrivals.

Gamma and Negative Binomial

There is a strong analogy:

Discrete	Continuous	Interpretation
Geometric	Exponential	waiting for first event
Negative Binomial	Gamma	waiting for multiple events

Big picture

Gamma distributions are continuous analogs of Negative Binomial distributions.

MGF of Gamma Distribution

If

$$X \sim \text{Gamma}(\alpha, \lambda),$$

then its MGF is

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha},$$

for $t < \lambda$.

Proof: MGF of Gamma Distribution

The density is

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0.$$

By definition,

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} f_X(x)\, dx.$$

Therefore,

$$M_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{tx} e^{-\lambda x}\, dx.$$

Combining the exponentials, we have

$$M_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x}\, dx.$$

Using the Gamma integral,

$$\int_0^\infty x^{\alpha-1} e^{-cx}\, dx = \frac{\Gamma(\alpha)}{c^\alpha}, \qquad c > 0,$$

with $c = \lambda - t$, we get

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \qquad \text{for } \lambda - t > 0.$$

If

$$X \sim \text{Gamma}(a, 1), \qquad Y \sim \text{Gamma}(b, 1),$$

independently, then

$$\frac{X}{X+Y} \sim \text{Beta}(a, b).$$

Beautiful connection

Beta and Gamma distributions are deeply linked.

Summary

- Beta distributions model random quantities between

$$0 \quad \text{and} \quad 1.$$

- Gamma functions generalize factorials.
- Gamma distributions generalize Exponential distributions.
- Gamma distributions arise from sums of independent Exponentials.
- Beta and Gamma distributions are closely connected.

Next lecture

Conditional expectation and prediction.