MATH 394: Probability I

Supplementary Notes

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1 Continuity of Probability

Motivation

In probability, we often describe complicated events as limits of simpler events. For example, suppose we have events

$$A_1, A_2, \ldots, A_n, \ldots$$

and we want to understand the probability of an event obtained by taking an infinite union or an infinite intersection. The key fact is that the probability function $\mathbb{P}(\cdot)$ behaves continuously under limits of certain sequences of events, in a precise mathematical sense. This is called the **continuity of probability**, which we will prove here. Before that, we revisit the probability axioms.

Probability Space

A *probability space* is a triple

$$(S, \mathcal{F}, \mathbb{P}),$$

where

(1) **Sample space.**

S is the set of all possible outcomes of the experiment.

(2) **Sigma-algebra.**

$\mathcal{F}$ is a collection of subsets of S, called *events*, satisfying:

(a)

$$S \in \mathcal{F}.$$

(b) If $A \in \mathcal{F}$, then

$$A^c \in \mathcal{F}.$$

(c) If

$$A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F},$$

then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

A collection satisfying these three properties is called a *σ-algebra* (or *sigma-field*).

(3) **Probability measure.**

$\mathbb{P} : \mathcal{F} \to [0, 1]$ is a function that assigns a probability to every event in $\mathcal{F}$. Such a function is called a *probability measure* and satisfies the following axioms.

(A1) **Nonnegativity:**

$$\mathbb{P}(A) \geq 0, \qquad \text{for every } A \in \mathcal{F}.$$

(A2) **Normalization:**

$$\mathbb{P}(S) = 1.$$

(A3) **Countable additivity:**

If

$$A_1, A_2, \ldots, A_n, \ldots$$

are pairwise disjoint events, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Increasing and Decreasing Sequences of Events

Definition 1 (Increasing sequence of events)**.** A sequence of events $(A_n)_{n\geq 1}$ is called **increasing** if

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots .$$

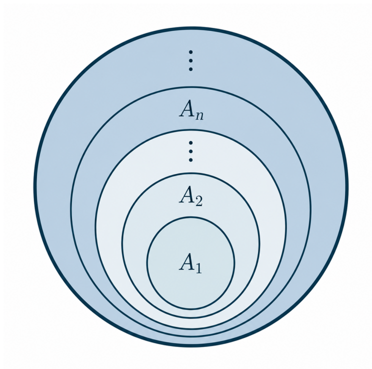

Figure 1: An increasing sequence of events, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, whose limit is $\bigcup_{n=1}^{\infty} A_n$.

In this case, we write

$$A_n \uparrow \bigcup_{n=1}^{\infty} A_n,$$

and we define

$$\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n.$$

So $A_n \uparrow \lim_{n\to\infty} A_n$ means that the events are getting larger and their limiting event is the union.

Definition 2 (Decreasing sequence of events)**.** A sequence of events $(A_n)_{n\geq 1}$ is called **decreasing** if

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots .$$

In this case, we write

$$A_n \downarrow \bigcap_{n=1}^{\infty} A_n,$$

where

$$\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n.$$

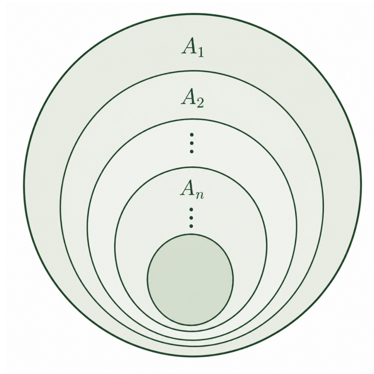

Figure 2: A decreasing sequence of events, $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, whose limit is $\bigcap_{n=1}^{\infty} A_n$.

So $A_n \downarrow \bigcap_{n=1}^{\infty} A_n$ means that the events are getting smaller and their limiting event is the intersection.

Continuity of the probability function from Below

Theorem 1 (Continuity from below)**.** *Suppose* $(A_n)_{n\geq 1}$ *is a sequence of events in* $\mathcal{F}$*, and*

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots .$$

Then

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Equivalently, if $A_n \uparrow A$*, then*

$$\mathbb{P}(A_n) \uparrow \mathbb{P}(A).$$

Proof. Define disjoint pieces

$$B_1 = A_1, \qquad B_n = A_n \setminus A_{n-1} \quad \text{for } n \geq 2.$$

These are disjoint as for any $m < n$, we have

$$B_n \cap B_m = (A_n - A_{n-1}) \cap (A_m - A_{m-1}) = A_n \cap A_{n-1}^c \cap A_m \cap A_{m-1}^c$$

Since $m < n$, we have $A_m \subseteq A_n$, and $A_{m-1} \subseteq A_{n-1}$, and therefore, $A_{n-1}^c \subseteq A_{m-1}^c$. This implies that $A_n \cap A_m = A_m$ and $A_{n-1}^c \cap A_{m-1}^c = A_{n-1}^c$. Now, since $n-1 \leq m$, $A_{n-1} \subseteq A_m$ and therefore, $A_m \cap A_{n-1}^c = \emptyset$. Hence, we have shown

$$A_n \cap A_{n-1}^c \cap A_m \cap A_{m-1}^c = A_m \cap A_{n-1}^c = \emptyset$$

So $B_n \cap B_m = \emptyset$.

Next, we claim (please show) that for every $n \geq 1$,

$$A_n = \bigcup_{k=1}^{n} B_k.$$

Also,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Therefore, by countable additivity,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(B_n).$$

On the other hand, for each fixed n,

$$\mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} \mathbb{P}(B_k).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(B_k) = \sum_{k=1}^{\infty} \mathbb{P}(B_k).$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

□

Remark 1 (Analogy with left-continuity of functions). Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous from the left at a point a. If

$$a_n \uparrow a,$$

meaning

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots \qquad \text{and} \qquad a_n \to a,$$

then

$$f(a_n) \to f(a).$$

Equivalently,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a).$$

Continuity from below for probability says something very similar. If

$$A_n \uparrow A,$$

then

$$\mathbb{P}(A_n) \to \mathbb{P}(A).$$

Equivalently,

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n)$$

In this sense, probability is continuous along increasing sequences of events. The role of "approaching from the left" is played by the inclusions

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} A_i.$$

Continuity of the probability function from Above

Theorem 2 (Continuity from above)**.** *Suppose* $(A_n)_{n\geq 1}$ *is a sequence of events in* $\mathcal{F}$*, and*

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots .$$

Then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Equivalently, if $A_n \downarrow A$*, then*

$$\mathbb{P}(A_n) \downarrow \mathbb{P}(A).$$

Proof. Let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Since $A_n \downarrow A$, the complements satisfy

$$A_1^c \subseteq A_2^c \subseteq \cdots \subseteq A_n^c \subseteq \cdots .$$

Also, by De Morgan's law,

$$\bigcup_{n=1}^{\infty} A_n^c = \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A^c.$$

By continuity from below,

$$\mathbb{P}(A^c) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n^c\right) = \lim_{n\to\infty} \mathbb{P}(A_n^c).$$

Using complements,

$$\mathbb{P}(A_n^c) = 1 - \mathbb{P}(A_n),$$

and

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Therefore,

$$1 - \mathbb{P}(A) = \lim_{n\to\infty} \left(1 - \mathbb{P}(A_n)\right).$$

Hence,

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

□

Remark 2 (Analogy with right-continuity of functions)*.* Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right at a point a. If

$$a_n \downarrow a,$$

meaning

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \qquad \text{and} \qquad a_n \to a,$$

then

$$f(a_n) \to f(a).$$

Equivalently,
$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a).$$

Continuity from above for probability says the analogous statement for decreasing sequences of events. If
$$A_n \downarrow A,$$
then
$$\mathbb{P}(A_n) \to \mathbb{P}(A).$$

Here, the role of "approaching from the right" is played by
$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \supseteq A.$$

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Properties of a Cumulative Distribution Function

Let X be a random variable on the probability space
$$(S, \mathcal{F}, \mathbb{P}).$$
The **cumulative distribution function**, or **CDF**, of X is the function
$$F : \mathbb{R} \to [0,1]$$
defined by
$$F(x) = \mathbb{P}(X \leq x).$$

More precisely,
$$F(x) = \mathbb{P}(\{\omega \in S : X(\omega) \leq x\}).$$

In fact, for any $B \subseteq \mathbb{R}$, we write $\{X \in B\}$ instead of $\{\omega \in S : X(\omega) \in B\}$.

Proposition 1 (Basic properties of a CDF)**.** *Let F be the CDF of a random variable X. Then:*

(1)
$$0 \leq F(x) \leq 1 \qquad \textit{for every } x \in \mathbb{R}.$$

(2)
$$\lim_{x\to-\infty} F(x) = 0.$$

(3)
$$\lim_{x\to\infty} F(x) = 1.$$

(4) F is increasing. That is, if $x \leq y$, then
$$F(x) \leq F(y).$$

Proof. We prove each property separately.

(1) Proof that $F(x) \in [0, 1]$.

Fix $x \in \mathbb{R}$. By definition,

$$F(x) = \mathbb{P}(X \le x).$$

The set $\{X \le x\}$ is an event in $\mathcal{F}$. Since $\mathbb{P}$ is a probability measure, every event has probability between 0 and 1. Therefore,

$$0 \le \mathbb{P}(X \le x) \le 1.$$

Hence,

$$0 \le F(x) \le 1.$$

(2) Proof that $\lim_{x \to -\infty} F(x) = 0$.

For each $n \in \mathbb{N}$, define

$$A_n = \{X \le -n\}.$$

As n increases, the number $-n$ becomes smaller. Therefore, the events become smaller, and thus, we have a decreasing chain of events.

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots.$$

Reason: If $\omega \in A_{n+1}$, then $X(\omega) \le -(n+1) < -n$, so $X(\omega) \le -n$. Thus, $\omega \in A_n$., and hence, $A_{n+1} \subseteq A_n$.

Now consider the intersection

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \{X \le -n\}.$$

This event means that X is less than or equal to $-n$ for every positive integer n. But no real number can be less than or equal to every number

$$-1, -2, -3, \ldots.$$

Since X is real-valued, this is impossible. Therefore,

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

By Theorem 2,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n).$$

Thus,

$$\mathbb{P}(\emptyset) = \lim_{n \to \infty} \mathbb{P}(X \le -n).$$

Since $\mathbb{P}(\emptyset) = 0$, we get

$$\lim_{n \to \infty} F(-n) = 0.$$

This shows that along the sequence $x = -n$,

$$F(x) \to 0.$$

Since F is increasing, this implies the full limit

$$\lim_{x \to -\infty} F(x) = 0.$$

(3) Proof that $\lim_{x \to \infty} F(x) = 1$.
For each $n \in \mathbb{N}$, define

$$B_n = \{X \leq n\}.$$

As n increases, the number n becomes larger. Therefore, the events become larger:

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots .$$

Reason: If $\omega \in B_n$, then $X(\omega) \leq n < n+1$,, so $X(\omega) \leq n+1$. Thus, $\omega \in B_{n+1}$.. Hence, $B_n \subseteq B_{n+1}$.
Now consider the union.

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \{X \leq n\}.$$

Since X is real-valued, for every outcome $\omega \in S$, the number $X(\omega)$ is a real number. Therefore, there exists some positive integer n such that

$$X(\omega) \leq n.$$

Hence every $\omega \in S$ belongs to at least one of the events B_n. Therefore,

$$\bigcup_{n=1}^{\infty} B_n = S.$$

By Theorem 1,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} \mathbb{P}(B_n).$$

Thus,

$$\mathbb{P}(S) = \lim_{n \to \infty} \mathbb{P}(X \leq n).$$

Since $\mathbb{P}(S) = 1$, we get

$$\lim_{n \to \infty} F(n) = 1.$$

This shows that along the sequence $x = n$,

$$F(x) \to 1.$$

Since F is increasing and bounded above by 1, this implies the full limit

$$\lim_{x \to \infty} F(x) = 1.$$

(4) Proof that F is increasing.
Let $x, y \in \mathbb{R}$ and suppose

$$x \leq y.$$

We claim that

$$\{X \leq x\} \subseteq \{X \leq y\}.$$

Indeed, if $\omega \in \{X \leq x\}$, then

$$X(\omega) \leq x.$$

Since $x \leq y$, we have

$$X(\omega) \leq y.$$

Therefore,

$$\omega \in \{X \leq y\}.$$

So,

$$\{X \leq x\} \subseteq \{X \leq y\}.$$

By monotonicity of probability,

$$\mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y).$$

Therefore,

$$F(x) \leq F(y).$$

Hence F is increasing. □